Some numerical methods

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A few examples of generating random numbers following given distributions are presented. In addition, a geometrical/numerical method to calculate the number π is presented.

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Outline:

- Generation of numbers following the Gaussian (normal) distribution using the central limit theorem.
- A direct way to generate numbers following the Gaussian (normal) distribution.
- Generation of numbers following distributions which are integrable and the integral is invertible.
- The distribution $1/x^4$

- A standard method to generate numbers following a given distribution.
- A simplistic way to calculate the number π

1 The normal distribution from the central limit theorem

- The sum of many numbers following some distribution approximates the normal distribution.
- We add successively more uniformly distributed numbers on the interval (0,1). Adding two such numbers, we obtain the characteristic triangle distribution, as shown in figure 1. Figure 2 shows the result from adding 12 uniformly distributed numbers.

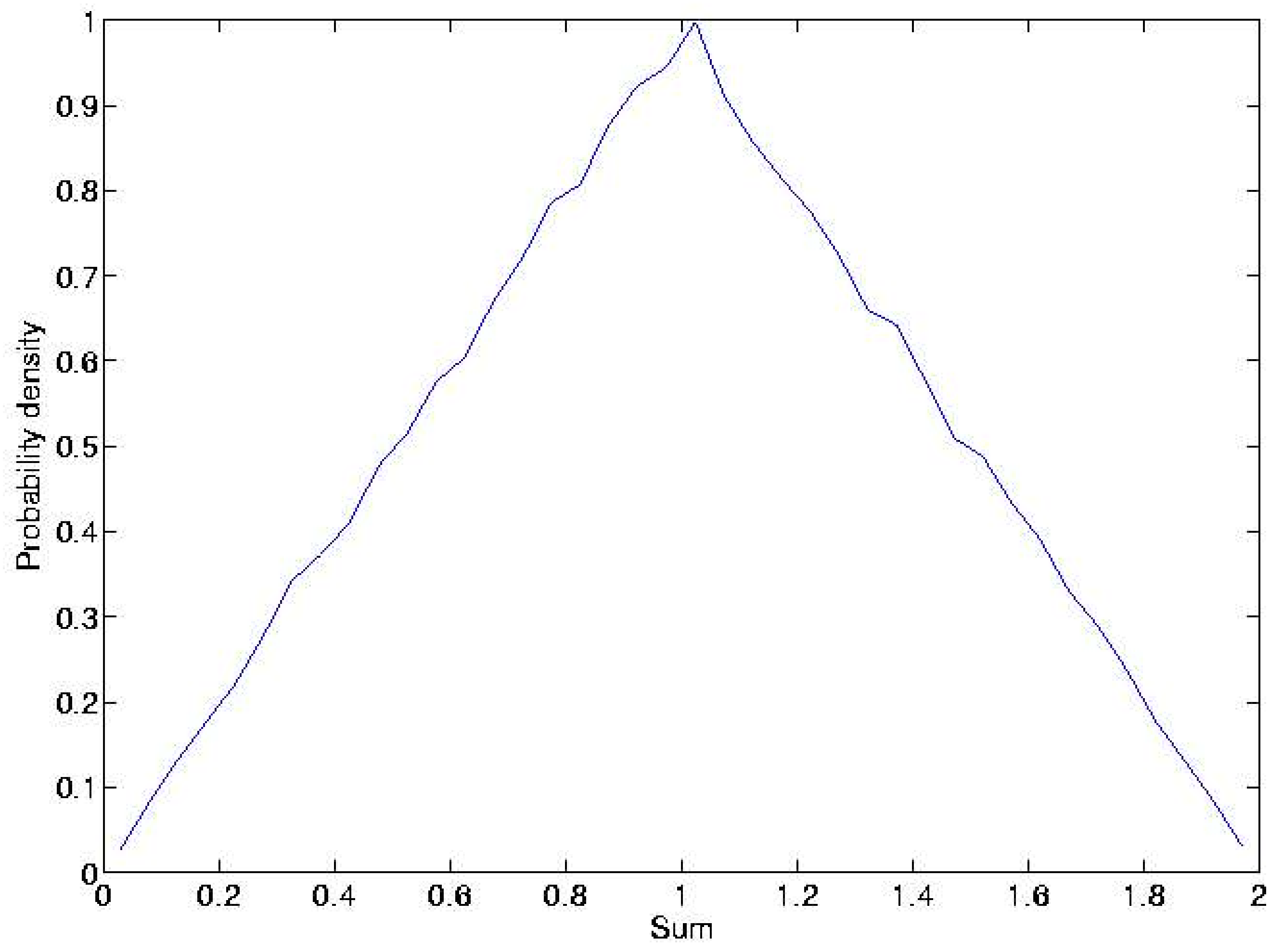

Figure 1: The triangle distribution obtained by adding two uniformly distributed numbers.

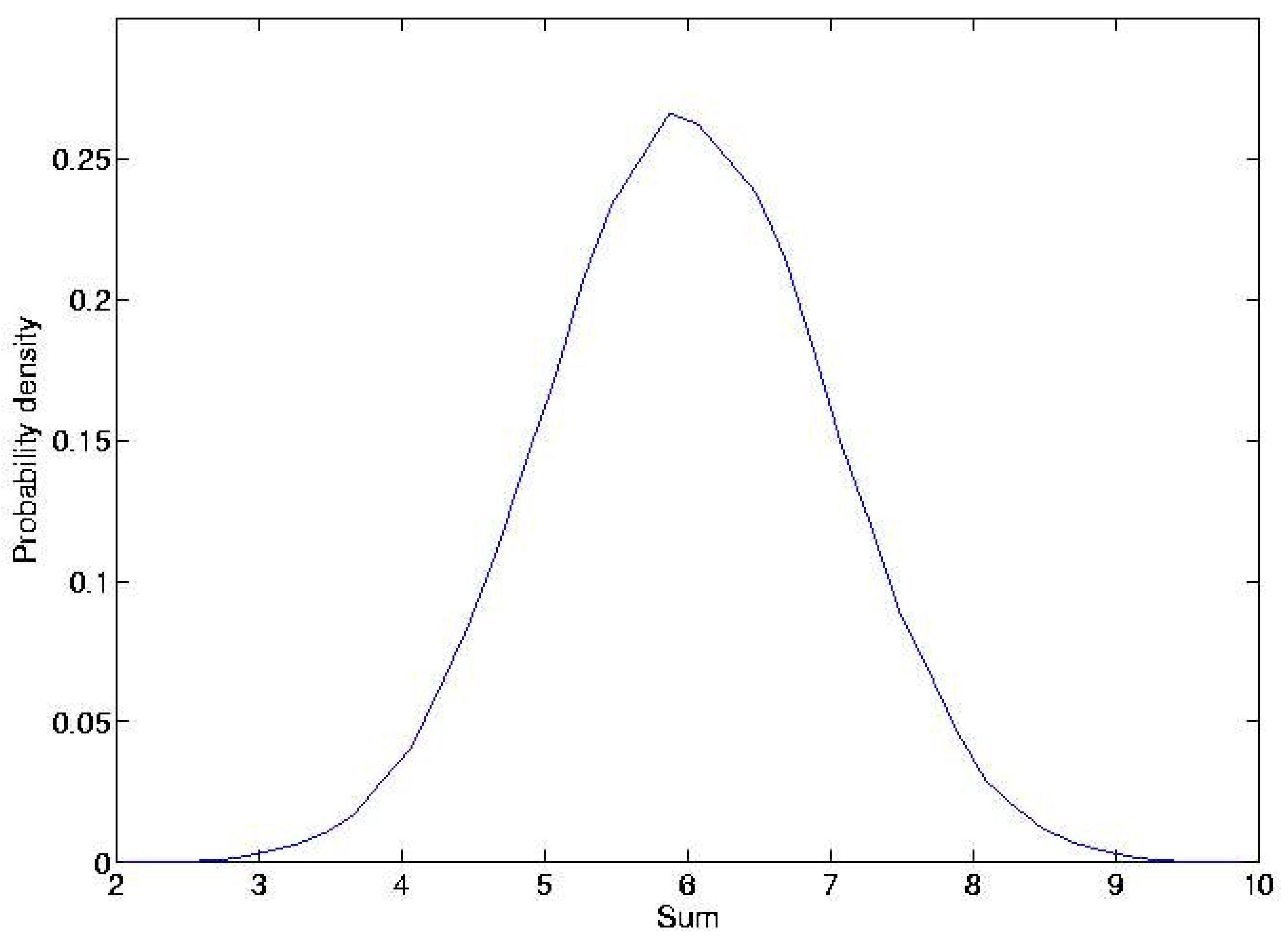

Figure 2: The approximate Gaussian distribution obtained by adding twelve uniformly distributed numbers.

2 A direct way to generate numbers following the Gaussian distribution

- Two independent numbers *X*, *Y*, following the normal distr. $(\mu, \sigma) = (0,1)$ are to be generated.
- That *X* and *Y* are independent and normal (0,1) means that the quadratic sum $r^2 = X^2 + Y^2$ is χ^2-distributed with two degrees of freedom. In polar coordinates $X = r\cos\theta$, $Y = r\sin\theta$. For the probability density, *f*, we thus write

$$f(r^2) = \frac{1}{2} e^{-r^2/2}$$

- The cumulative probability is uniformly distributed:

$$F(r^2) = \int_0^{r^2} \frac{1}{2} e^{-r'^2/2} dr'^2 = 1 - e^{-r^2/2} \equiv u \text{ uniform on } (0,1)$$

- From the equation $1 - e^{-r^2/2} = u$ we obtain

$$r = \sqrt{-2\ln(1-u)}$$

- We generate θ uniformly over $(0, 2\pi)$ and u uniformly over (0,1). Then $X = r\cos\theta$, $Y = r\sin\theta$.

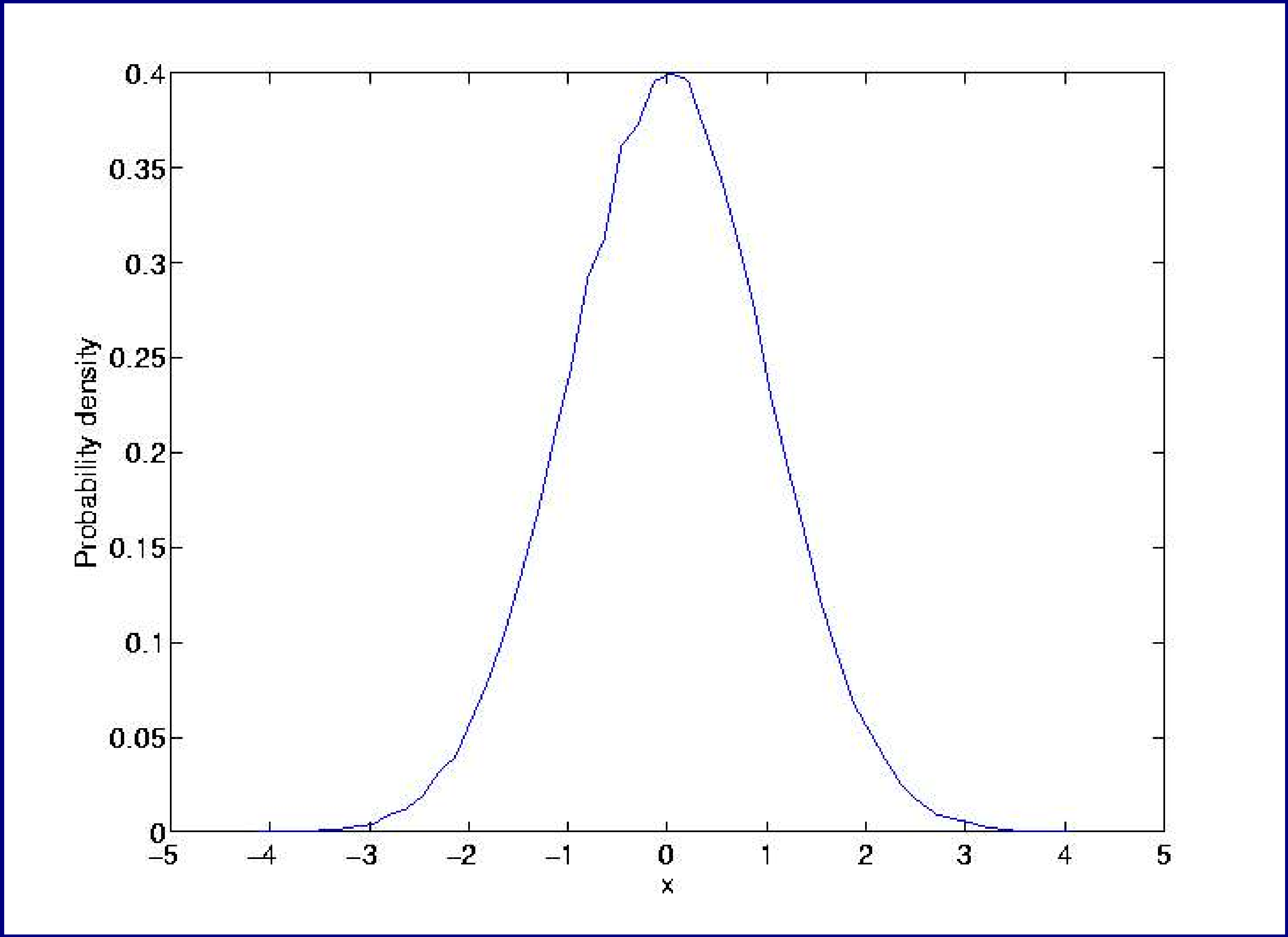

Figure 3: The one-dimensional Gaussian distribution X.

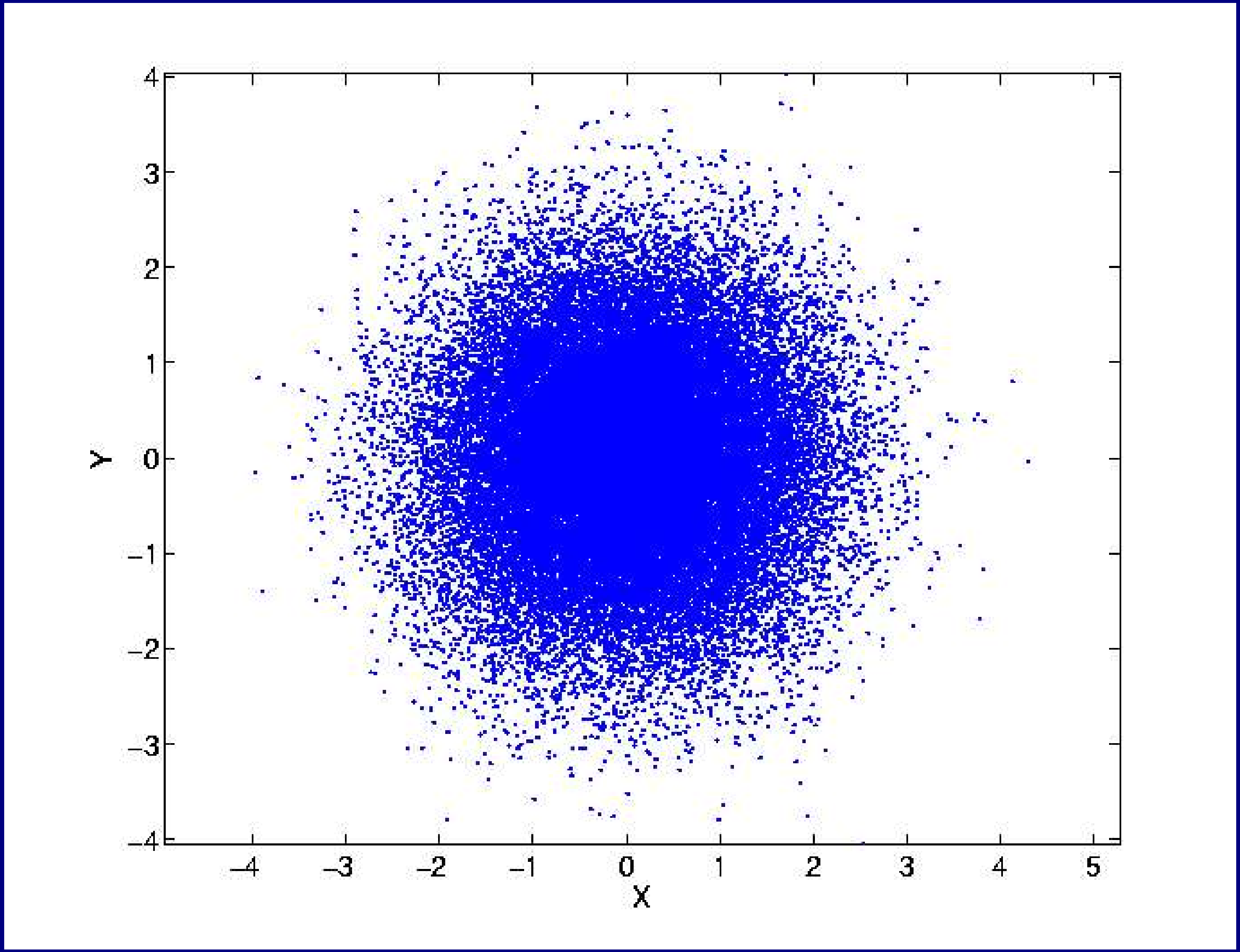

Figure 4: The two-dimensional Gaussian distribution Y vs X.

3 Generation of numbers following distributions which are integrable and the integral is invertible

- Let X be a random variable with probability density $f(x)$.
- We define the cumulative probability function $F(x)$ by

$$F(x) = Pr(X \leqslant x) = \int_{-\infty}^{x} f(x')\,dx'$$

- $Y = F(X)$ is then uniformly distributed, i.e. the probability density of $F(X)$ equals unity on (0,1).

- Observe that $X = F^{-1}(Y)$.
- Generating X is then done by generating Y uniformly on the interval (0,1), and applying F^{-1}.

The distribution $1/x$ in some detail

- We want to generate X with probability density proportional to $1/x$ on (1,10).
- We define the normalization constant C as

$$C = \int_1^{10} \frac{1}{x} dx = \ln 10 - \ln 1 = \ln 10$$

- We want to generate X with probability density

$$f(x) = \frac{1}{C}\frac{1}{x} \quad \text{for } x \text{ on } (1,10), \text{ 0 otherwise}$$

- Then

$$F(x) = \int_{-\infty}^{x} f(x')\,dx' = \frac{1}{C}\int_{1}^{x} \frac{1}{x'}\,dx' = \frac{1}{C}\ln x$$

- The inverse function is $F^{-1}(x) = 10^x$
- We generate Y uniformly on (0,1), and apply F^{-1} to obtain X. The result is shown in figure 5.

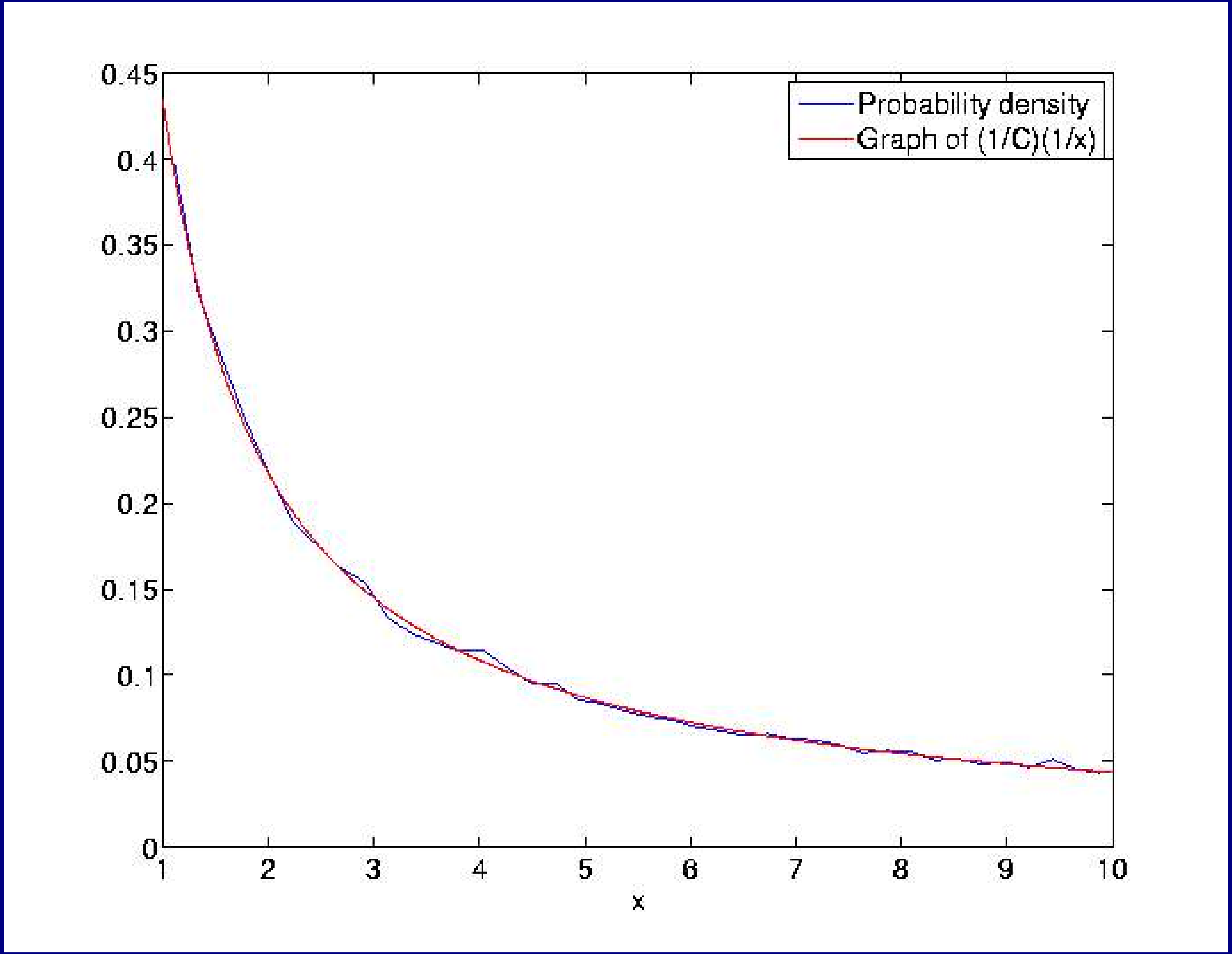

Figure 5: The distribution $1/(Cx)$ plotted together with the true graph of $1/(Cx)$.

The distribution $1/x^4$

- The angular distribution of elastic electron-positron scattering (Bhabha scattering) is approximately proportional to $1/\theta^4$ for small scattering angles θ.
- We generate $1/(Cx^4)$ on (5/1000,50/1000) (radians) using the same technique as in the previous example. The result is shown in figure 6.

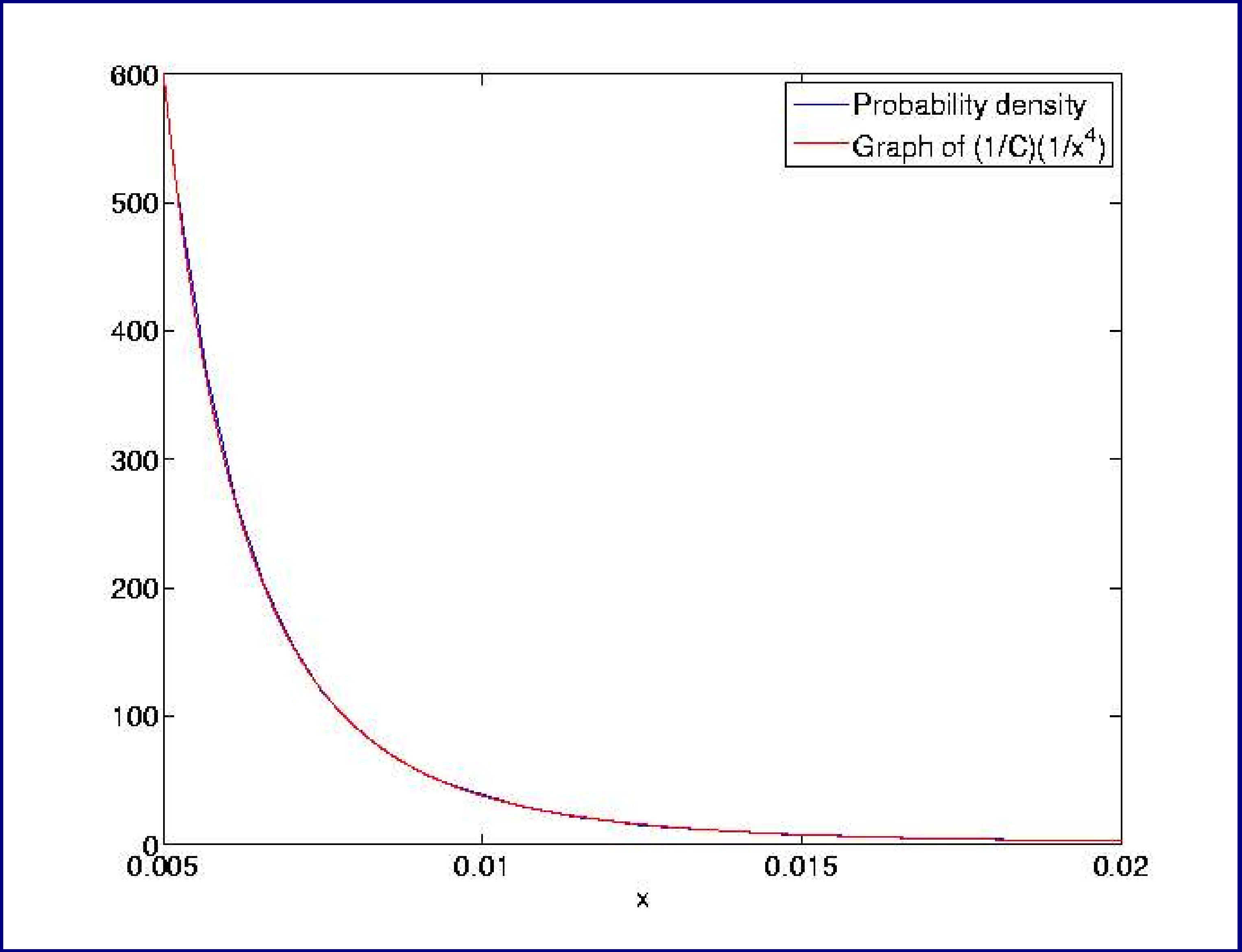

Figure 6: The distribution $1/(Cx^4)$ plotted together with the true graph of $1/(Cx^4)$.

4 A standard method to generate numbers following a given distribution

- A method is presented for generating random numbers following a probability density $f(x)$ on (a,b)
- The probability density can be in either analytical or histogram form
- We define y_{max} to be greater than or equal to the max value of $f(x)$ on (a,b)

- X is generated uniformly over (a,b). For given X, Y is generated uniformly from 0 to y_{max} .
- If $Y < f(X)$ X is accepted, otherwise rejected
- The resulting X follows the distribution $f(x)$
- As an example, X with probability density function $f(x) = (1/2)\sin(x)$ for x in $(0,\pi)$ was generated. Result in figure 7.
- This method is more general than the one in section 3, but not nearly as fast

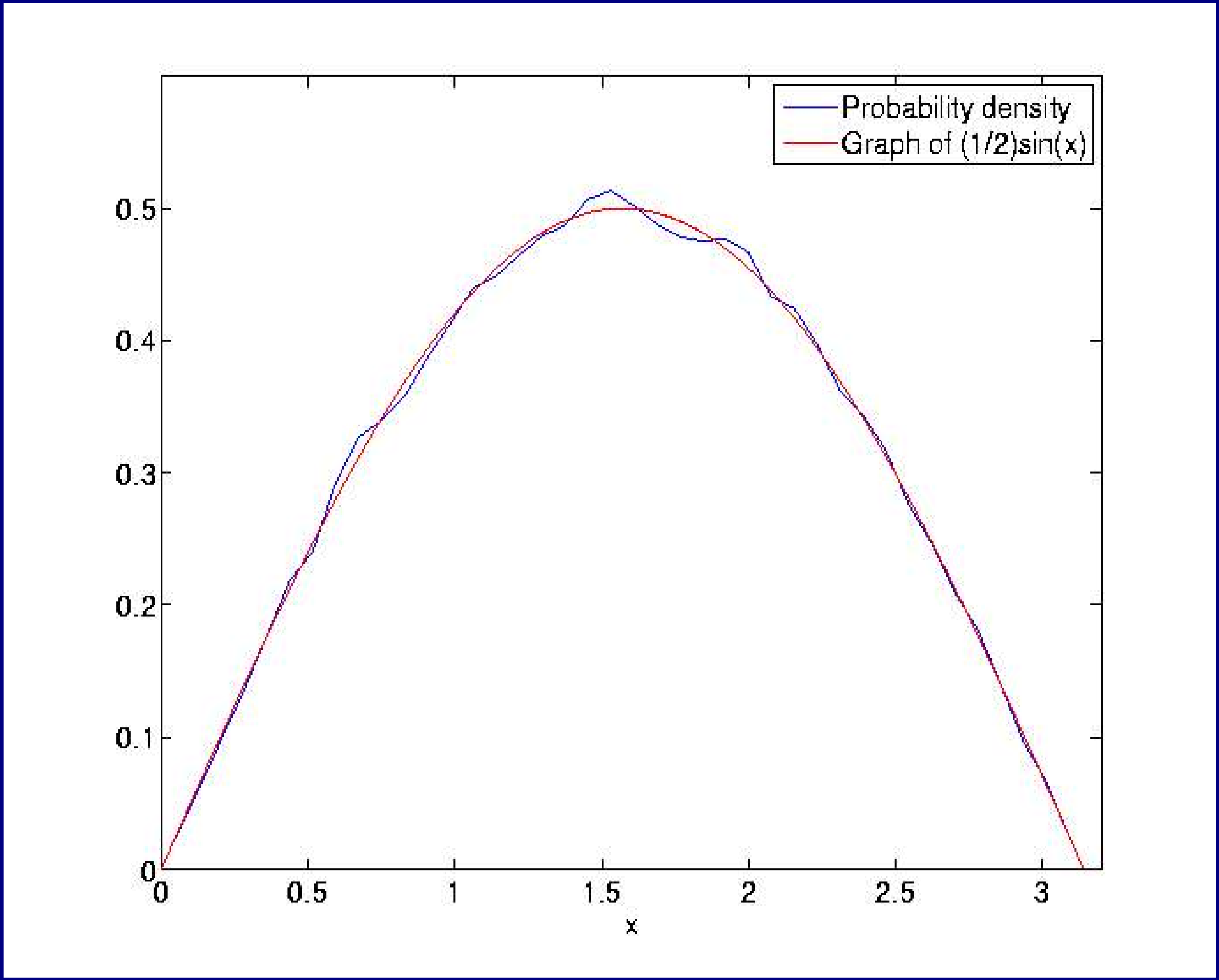

Figure 7: The distribution (1/2)sin(x) plotted together with the true graph of (1/2)sin(x).

5 A simplistic way to calculate the number π (geometrically inspired)

- Consider N_{gen} points (x,y) randomly generated over a square with sides of length 2.
- Inscribed in the square is a unit circle (radius 1).
- The number of points falling inside the circle, N_{acc}, is counted.
- The estimated ratio of the areas of the circle and the square is $\pi/4$:

$$\frac{\pi}{4} = \frac{(\text{Area of unit circle})}{(\text{Area of square})} \approx \frac{N_{\text{acc}}}{N} \Rightarrow \pi \approx 4\frac{N_{\text{acc}}}{N}$$

- This method was applied with 100 000 points generated.
- The generated points are shown in figure 8, the accepted ones in figure 9.
- From this, a value of 3.1390 was estimated for π
- A better approximation can be obtained by generating more points (requiring more calculation time).

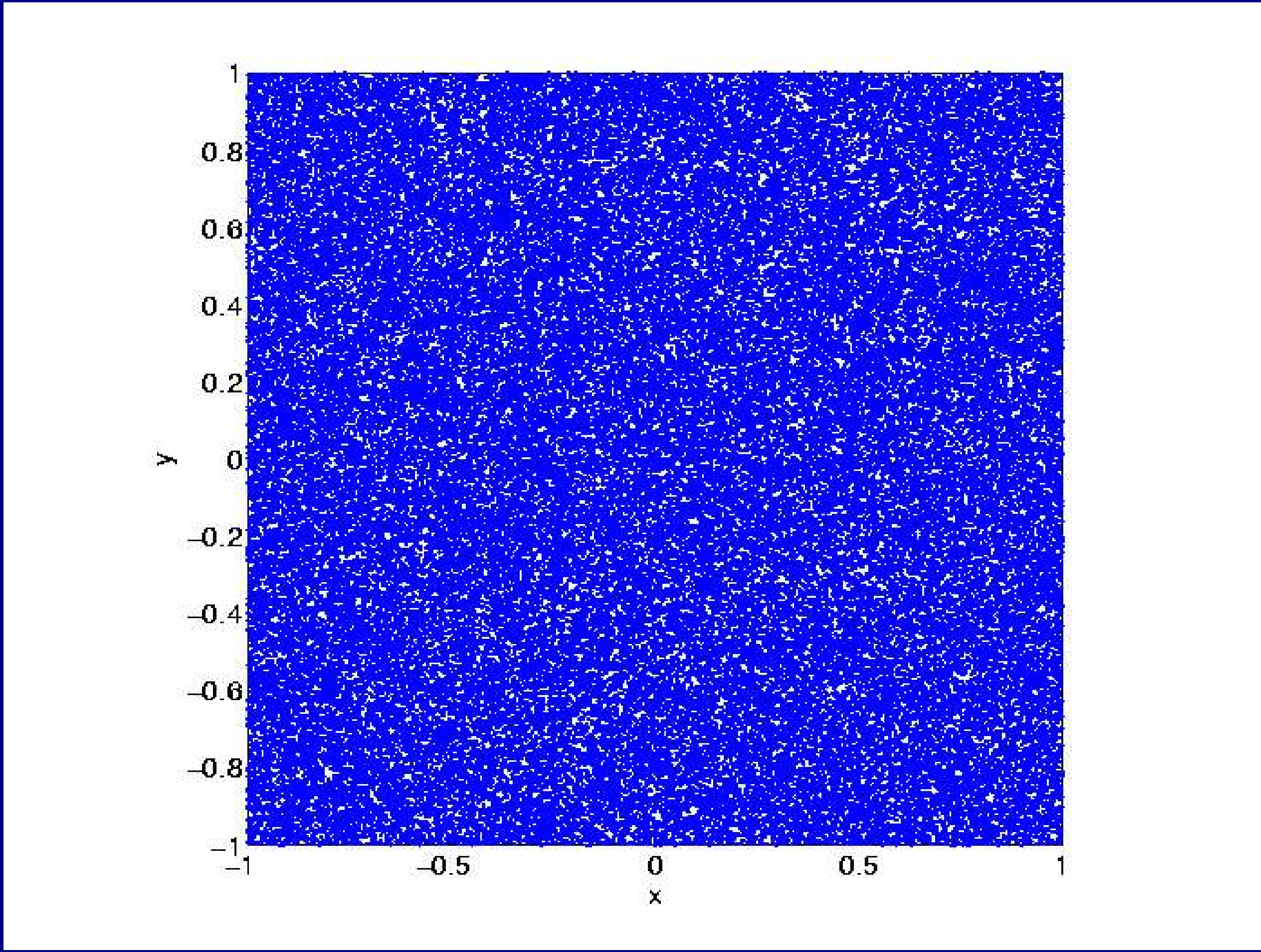

Figure 8: Points (x, y) generated uniformly over the square.

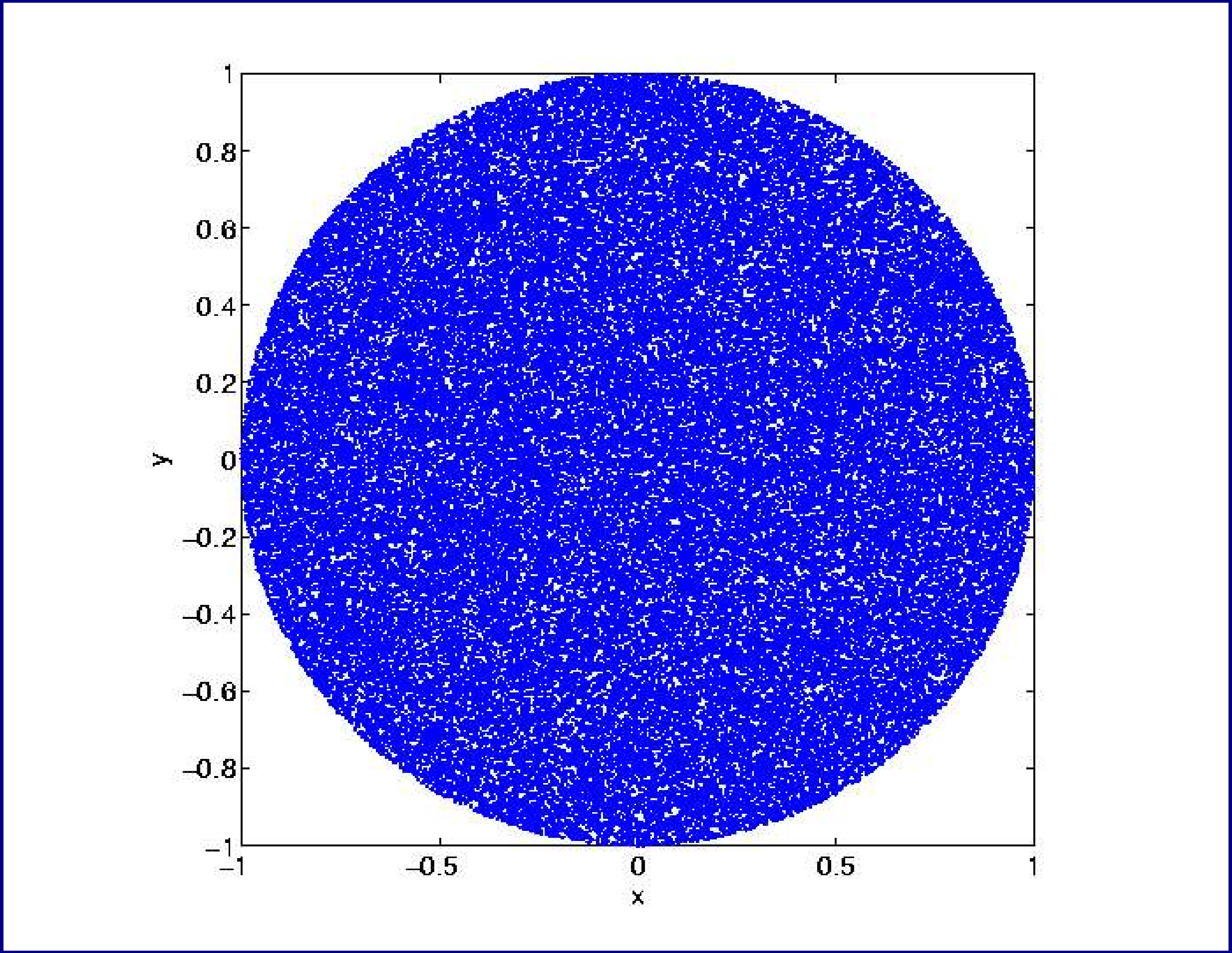

Figure 9: Points (x,y) inside the unit circle.