A new formulation of the nuclear level density by spin and parity

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Introduction

Level densities are important for nuclear reaction calculations. They are typically calculated with the Constant Temperature (CT) and Back Shifted Fermi Gas (BSFG) formulations. Each formulation has two parameters that are compiled in the IAEA Reaction Input Parameter Library (RIPL).

• Both models calculate total level densities independent of spin/parity.
• Spin distribution is calculated with a spin distribution function which is dependent on the spin cutoff parameter $\sigma_c$.
• Parity is not calculated.

All reaction experiments populate a subset of spins and parities.

• The spin distribution function is invalid at low excitation energy.
• Level density parameters and $\sigma_c$ are calculated using low energy level data.
• At low energies for most nuclei the levels are predominantly one parity.
• Comparing reaction data to RIPL total level densities is invalid.

Today I will discuss a modification of the CT formulation that is

• Valid for all spins and parities
• Applicable to nuclei with no resonance data
Level Density Formulations

### Constant Temperature (CT)

\[
N(E) = \exp\left[\frac{(E - E_0)}{T}\right]
\]

\[
\rho(E, J) = \frac{f(J) \cdot N(E)}{T}
\]

- \(E\) – level energy
- \(N(E)\) – level sequence number
- \(E_0\) – back shift parameter
- \(T\) – critical energy for breaking nucleon pairs

T. Ericson NP 11, 481 (1959)

### Back Shifted Fermi Gas (BSFG)

\[
\rho(E, J) = f(J) \frac{\exp\left[2\sqrt{a(E - E_1)}\right]}{12\sqrt{2}\sigma_c a^{0.25}(E - E_1)^{1.25}}
\]

- \(a\) – level density parameter
- \(E_1\) – back shift parameter
- \(\sigma_c\) – spin cutoff parameter


### Spin Distribution Function

\[
f(J) = \frac{2J + 1}{2\sigma_c^2} \exp\left[-\frac{(J + \frac{1}{2})^2}{2\sigma_c^2}\right]
\]


\(E_0, T, a, E_1, \sigma_c\) are parameters, see T. von Egidy and D. Bucurescu, Phys. Rev. C 72, 044311 (2005) and RIPL-3.

Calculation of these parameters is described by Gilbert and Cameron, Can. J. Phys. 43, 1446 (1965).

Other models take into account shell and collective effects or apply combinatorial methods.
CT Formulation

\[ N(E) = \exp \left[ \frac{(E - E_0)}{T} \right] \]

\[ \rho(E, J) = \frac{f(J) \cdot N(E)}{T} \]

For the rest of this talk I will discuss the CT formulation.

Basic assumptions

- The total level density increases exponentially with a temperature \( T \) and a back shifted level energy \( (E - E_0) \).
- The level density for each spin has the same \( T \) and \( E_0 \).
- \( T \) is a fundamental parameter that can be accurately described, for example, as the BCS critical temperature (Moretto et al, Journal of Physics: Conference Series 580 (2015) 012048) for \( A > 100 \).
  \[ T_{CR} = \frac{2\Delta_0}{3.53}, \quad \Delta_{BM} \approx 12A^{-\frac{1}{2}}, \quad T \approx 6.80A^{-\frac{1}{2}} \]
  \( \Delta_{BM} \) is the gap parameter from the liquid drop model (Bohr and Mottelson)
- \( E_0 \) is a meaningless fitting parameter
- Spin is a separable function \( f(J) \).
- Parity is ignored.
- The useful level energy range of the CT formulation is not apparent.
How are the CT parameters derived?

The level number $N(E_{S_n})$ at the neutron separation energy is

$$N(E_{S_n}) = \frac{T}{D_0 \cdot f(1/2^+)}$$

where $D_0$ is the s-wave n-capture level spacing and $f(1/2^+) = 0.5 \cdot f(1/2)$.

From the spin distribution function.

$T$ and $E_0$ are derived solely from an exponential fit to the low-lying levels. $T$ is independent of $E_0$.

Level sequence numbers increase exponentially for all nuclei. This is a fundamental observation of RIPL.
Level spacing increases exponentially for all $J^\pi$

Selected spins can also be fit to an exponential.

Bandhead energies can be fit to ±108 keV, $\chi^2/f=1.0$
CT-JPI Formulation

- Define $E_0$ as the Yrast energy for each spin and parity.
  
  *This requires $N(E)=1$ for the first occurrence of each $J^\pi$ value. Natural value*

- $T$ is a constant temperature fit to all spin and parity sequences
  
  *This is a primary tenet of the CT formulation*

- $T(J^\pi)_{exp}$ and $D(J^\pi)_{exp}$, the level spacing at $S_n$ for $J^\pi$, are determined by an exponential fit of $N(E)$ to $E-E_0(J^\pi)$ for the resonance energies $E$ of each $J^\pi$.

- $D(J^\pi)_{calc}$ is determined from the spin distribution function for $J^\pi$ when no resonance data exists where the spin cutoff parameter $\sigma_c$ is chosen so that $D(J^\pi)_{calc}$ coincides with experiment.

  *This formulation provides a self consistent level density calculation requiring only one free parameter, $T$.*

The CT-JPI formulation differs from the CT formulation in that $E_0$ is a physical parameter and $T$ is determined from resonance data rather than level data.
The CT-JPI Formulation – $^{57}$Fe

Experimental Resonance Data (292 s-, p-, d-wave Resonances)

CT-JPI fit to experimental resonance data for $^{57}$Fe. The negative parity levels have constant $T$, positive parity levels have variable $T$.

RIPL: $T=1.31$ MeV, $E_0=-1.41$ MeV
The CT-JPI Formulation – $^{57}$Fe

**Spin Distribution Calculations**

$\sigma_c = 2.56$

$\sigma_c$ was varied to get constant $T$

$\sigma_c = 3.17$ (von Egidy)

<table>
<thead>
<tr>
<th>$J^\pi$</th>
<th>$E_0$ [10] MeV</th>
<th>$T(J^\pi)$ MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^+$</td>
<td>3.2983</td>
<td>1.231(5)</td>
</tr>
<tr>
<td>$3/2^+$</td>
<td>2.8359</td>
<td>1.132(4)</td>
</tr>
<tr>
<td>$5/2^+$</td>
<td>2.5050</td>
<td>1.27(14)</td>
</tr>
<tr>
<td>$7/2^+$</td>
<td>3.100</td>
<td>1.13</td>
</tr>
<tr>
<td>$9/2^+$</td>
<td>2.4558</td>
<td>1.38</td>
</tr>
<tr>
<td>$11/2^+$</td>
<td>3.8747</td>
<td>1.25</td>
</tr>
<tr>
<td>$3/2^-$</td>
<td>0.0</td>
<td>1.685(5)</td>
</tr>
<tr>
<td>$5/2^-$</td>
<td>0.1364</td>
<td>1.66</td>
</tr>
<tr>
<td>$7/2^-$</td>
<td>1.0072</td>
<td>1.57</td>
</tr>
<tr>
<td>$9/2^-$</td>
<td>1.1984</td>
<td>1.68</td>
</tr>
<tr>
<td>$11/2^-$</td>
<td>2.3563</td>
<td>1.66</td>
</tr>
</tbody>
</table>

Shaded values from experiment, others from spin distribution function. I get three temperatures averaging all values.

$T=1.664$ MeV, All $J$, $\pi = -$  
$T=1.131$ MeV, $J=1/2^+, 5/2^+, 9/2^+, ...$

$T=1.28$ MeV, $J=3/2^+, 7/2^+, 11/2^+, ...$

$$D(J^\pi) = \frac{f(J^\pi_{\text{exp}})}{f(J^\pi)} D(J^\pi_{\text{exp}}).$$
The CT-JPI Formulation – $^{57}$Fe

For higher spins $T$ and $D(J^\pi)$, the level spacing at $S_n$ for levels of $J^\pi$, calculated from the spin distribution function diverge.


<table>
<thead>
<tr>
<th>$J^\pi$</th>
<th>$T$(calc) MeV</th>
<th>$D(J^\pi)$(calc) keV</th>
<th>$T$(syst) MeV</th>
<th>$D(J^\pi)$(syst) keV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$13/2^+$</td>
<td>1.73</td>
<td>143</td>
<td>1.13</td>
<td>24</td>
</tr>
<tr>
<td>$15/2^+$</td>
<td>1.61</td>
<td>373</td>
<td>1.28</td>
<td>200</td>
</tr>
<tr>
<td>$17/2^+$</td>
<td>2.97</td>
<td>1278</td>
<td>1.13</td>
<td>74</td>
</tr>
<tr>
<td>$19/2^+$</td>
<td>3.24</td>
<td>4908</td>
<td>1.28</td>
<td>970</td>
</tr>
<tr>
<td>$21/2^+$</td>
<td>10.7</td>
<td>22044</td>
<td>1.13</td>
<td>364</td>
</tr>
<tr>
<td>$13/2^-$</td>
<td>1.91</td>
<td>164</td>
<td>1.66</td>
<td>74</td>
</tr>
<tr>
<td>$15/2^-$</td>
<td>2.50</td>
<td>450</td>
<td>1.66</td>
<td>202</td>
</tr>
<tr>
<td>$17/2^-$</td>
<td>3.70</td>
<td>1463</td>
<td>1.66</td>
<td>658</td>
</tr>
</tbody>
</table>

The spin distribution function fails for $J^\pi>11/2$. A constant temperature works better for high spin but also $\sigma_c$ could be varied.
Total level density – $^{57}$Fe

The total level density is determined by summing over all $J^\pi$ values. Agreement with RIPL CT calculation and Oslo data are good.

$\pi^+/\pi^- = 1.2$ at 9 MeV.
The CT-JPI Formulation – $^{236}$U

932 Resonances

D(3)/D(4) = 1.51 (expt), = 1.56 (calc)

$\sigma_c = 2.40$

$\sigma_c = 4.74$ (von Egidy)

<table>
<thead>
<tr>
<th>$J^\pi$</th>
<th>$E_0^{\alpha}$ [13] (MeV)</th>
<th>$T(J^\pi)$ (MeV)</th>
<th>$D(J^\pi)$ eV</th>
<th>$N(S_n, J^\pi)^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3$^-$</td>
<td>0.7442</td>
<td>0.456</td>
<td>1.37</td>
<td>333416</td>
</tr>
<tr>
<td>4$^-$</td>
<td>1.0525</td>
<td>0.421</td>
<td>0.90</td>
<td>467410</td>
</tr>
</tbody>
</table>
The CT-JPI Formulation – $^{236}$U

**Spin distribution function fit**

<table>
<thead>
<tr>
<th>$J^\pi$</th>
<th>$E_0^a$ [13] (MeV)</th>
<th>$T(J^\pi)$ (MeV)</th>
<th>$D(J^\pi)$ (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^-$</td>
<td>1.310</td>
<td>0.443</td>
<td>3.38</td>
</tr>
<tr>
<td>$1^-$</td>
<td>0.6876</td>
<td>0.460</td>
<td>1.33</td>
</tr>
<tr>
<td>$2^-$</td>
<td>0.9877</td>
<td>0.433</td>
<td>1.13</td>
</tr>
<tr>
<td>$3^-$</td>
<td>0.7442</td>
<td>0.456</td>
<td>1.37</td>
</tr>
<tr>
<td>$4^-$</td>
<td>1.0525</td>
<td>0.421</td>
<td>0.90</td>
</tr>
<tr>
<td>$5^-$</td>
<td>0.8481</td>
<td>0.456</td>
<td>1.75</td>
</tr>
<tr>
<td>$6^-$</td>
<td>1.164</td>
<td>0.461</td>
<td>2.33</td>
</tr>
<tr>
<td>$0^+$</td>
<td>0.9191</td>
<td>0.475</td>
<td>3.38</td>
</tr>
<tr>
<td>$1^+$</td>
<td>0.9579</td>
<td>0.440</td>
<td>1.34</td>
</tr>
<tr>
<td>$2^+$</td>
<td>0.9603</td>
<td>0.435</td>
<td>1.14</td>
</tr>
<tr>
<td>$3^+$</td>
<td>1.0015</td>
<td>0.438</td>
<td>$\equiv 1.37$</td>
</tr>
<tr>
<td>$4^+$</td>
<td>1.0508</td>
<td>0.421</td>
<td>$\equiv 0.90$</td>
</tr>
<tr>
<td>$5^+$</td>
<td>1.0938</td>
<td>0.438</td>
<td>1.75</td>
</tr>
</tbody>
</table>

**Assumption:** $D(3^-,4^-) = D(3^+,4^+)$.  
$\bar{T} = 0.443(5)$ MeV  
$T = 0.44(1)$, von Egidy  
$T = 0.393$ (RIPL)  
$T_{BCS} = 0.443$ MeV

**Note:** The Yrast GS band gives $T = 0.516(14)$ MeV. The Yrare band gives $T = 0.443(16)$ MeV and is used in the CT-JPI fit.

Spin distribution function fails at $J \geq 6$
The average level spacing is expected to follow a Wigner distribution

\[ P \left( \frac{S}{D} \right) = \frac{\pi}{2D} S \exp \left[ -\frac{\pi}{4} \frac{S^2}{D} \right], J^\pi_N = J^\pi_{N+1} \]

\[ P \left( \frac{S}{D} \right) = \exp \left( -\frac{S}{D} \right), J^\pi_N \neq J^\pi_{N+1} \]

where \( S = E^{J,\pi}_{N+1} - E^{J,\pi}_N, \quad D = 1/\rho(E,J,\pi) \)

Small variations in \( T \) are due to Wigner fluctuations in the positions of the Yrast levels. Assuming \( T=0.443 \text{ MeV} \), the effective Yrast energies can be calculated. Fluctuations in the Yrast energies, \( E_{\text{yrast}} - E_{\text{eff}} \), can be compared with the Wigner distribution. Reasonable agreement is obtained for \(^{236}\text{U}\).

Fluctuations in level spacing give modest agreement with Wigner distribution (missing levels?).

The validity of the Wigner distribution near the GS has been shown by Brody et al, Rev. Mod. Phys. 54, 385 (1981).
The CT-JPI Formulation – $^{236}$U

Level densities, $\pi^-/\pi^+ = 1.33$ at 6.5 MeV.
Improved $^{236}\text{U}$ Yrast Data

• **0$^-$ level**

  No 0$^-$ levels are known in $^{236}\text{U}$. The only 0- levels known in the actinides are at 1311.51 keV ($^{236}\text{Pu}$) and 1410.75 keV ($^{238}\text{Pu}$). Assuming $T=0.443(5)$ MeV, $E(0^-)=1330\pm140$ keV.

• **1$^+$ level**

  The Yrast 1$^+$ level from ENSDF at 1791.3 keV gives a low $T=0.380$ MeV. Levels at 957.9- and 960.3-keV were assigned (2$^+$) in ENSDF which is improbable for a Wigner distribution. The 957.9-keV level is not populated in average resonance capture from $J^\pi=3^-,4^-$. This level is consistent with an Yrast 1$^+$ assignment and gives $T=0.440$ MeV.

• **5$^+$ Level**

  Lowest candidate state is at 1093.8 keV with $J^\pi=(2,5)^+$ in ENSDF. Level is not feed by $^{236}\text{Pa} \beta^-$ decay (1$^-$) suggesting the higher spin is likely and gives $T=0.438$ MeV.
Conclusions

- The CT-JPI formulation can be used to calculate complete level schemes up to at least the neutron separation energy.

  This is a fundamental property of all nuclear level schemes

- The $E_0$ parameter restricts the $J^\pi$ level density to levels above the Yrast level.

  The parity distribution comes naturally from this formulation

- The temperature $T$ derived from resonance and low energy data are the same.

  The CT-JPI model is valid at all excitation energies

- The spin cutoff parameter, $\sigma_c$, is directly determined.

- The spacing of all levels follows a Wigner distribution.

- Calculation of the level density for high spin states is uncertain.

- More nuclei need to be studied to determine the range of applicability of the CT-JPI formulation.

**Epiphany:** For nuclei without resonance data the temperature can be derived from the low levels and combined with the Yrast data to determine the level density.
The experimental level spacing at $S_n$ is determined from the CT-JPI model parameters.

Agreement with the spin distribution function is good for $\sigma_c=3.98$. No $1^+$ or $0^-$ levels are predicted below $S_n=6154.3$ kev.
Thank you for your attention

Physics is much too hard for physicists.

— David Hilbert —