Susceptibility of superconductor disks and rings with and without flux creep

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First some consequences of the Bean assumption of constant critical current $J_c$ in type-II superconductors are listed and the Bean ac susceptibility of narrow rings is derived. Then flux creep is described by a nonlinear current-voltage law $E \propto J^p$, from which the saturated magnetic moment at constant ramp rate $H_a(t)$ is derived for rings with general hole radius $a_1$ and general creep exponent $n$. Next the exact formulation for rings in a perpendicular applied field $H_a(t)$ is presented in the form of an equation of motion for the current density in thick rings and disks or the sheet current in thin rings and disks. This method is used to compute general magnetization curves $m(H_a)$ and ac susceptibilities $\chi$ of rings with and without creep, accounting also for nonconstant $J_c(B)$. Typical current and field $B$ profiles are depicted. The initial slope of $m(H_a)$ (the ideal diamagnetic moment) and the field of full penetration are expressed as functions of the inner and outer ring radii $a_1$ and $a$. A scaling law is derived which states that for arbitrary creep exponent $n$ the complex nonlinear ac susceptibility $\chi(H_0, \omega)$ depends only on the combination $H_0^{1/n}/\omega$ of the ac amplitude $H_0$ and the ac frequency $\omega/2\pi$. This scaling law thus connects the known dependencies $\chi=\chi(\omega)$ in the ohmic limit ($n=1$) and $\chi=\chi(H_0)$ in the Bean limit ($n \to \infty$). [S0163-1829(97)01921-8]

I. INTRODUCTION

The magnetic moment $m = \frac{1}{2} \int r \times J(r) d^3r$ of a type-II superconductor is often calculated within the Bean model \cite{1,2} which assumes that the current density $J(r)$ cannot exceed a critical value $J_c$, $J=|J| \leq J_c$. If $J$ locally exceeds $J_c$, the flux lines will rearrange such that $J=J_c$ again holds in the entire superconductor. An additional assumption disregards the reversible magnetization, or the finite lower critical field $H_{c1}$, by writing $B=\mu_0 H$ inside (and outside) the material. Within this model, $m$ depends only on the applied magnetic field $H_a$ and on the geometry, i.e., on the sample shape and on the orientation of the applied field $H_a$. Being irreversible, $m$ depends on the history of $H_a(t)$, but it does not depend explicitly on the time $t$. This means the rate $\dot{H}_a(t)$ at which the external field is increased or decreased does not enter the Bean model.

In experiments on high-$T_c$ superconductors (HTSC’s) the effects of flux creep are often not negligible. For example, when $H_a(t)$ is held constant, the magnetization may relax and will thus depend on the time $t$. As a consequence, when $H_a(t)$ is cycled, the predictions of the static Bean model may be observed at very slow sweep rates, while at high sweep frequencies deviations occur due to flux creep. This finding may appear counterintuitive since creep effects are expected to be more visible after longer time, i.e., at slow sweep rates. Our quantitative theory below explains this observation in terms of a scaling law which states that the same ac susceptibility is obtained at higher sweep frequency if the sweep amplitude is increased accordingly. Within continuum theory, flux creep effects originate from the nonzero resistivity of HTSC’s occurring at sufficiently high temperatures and fields.

In this paper I investigate the influence of flux creep on the magnetization curves and nonlinear ac susceptibilities of superconducting disks and rings in a perpendicular field. In Sec. II some general features of the static Bean model are listed. The magnetization curves following from the Bean model for some basic longitudinal and transverse geometries are compiled in Sec. III. The nonlinear ac susceptibility of a narrow ring is given in Sec. IV. Flux creep effects and the saturation magnetization are discussed in Sec. V. Section VI presents exact equations of motion for the current density in thick disks and rings and for the sheet current in thin disks and rings; it is shown how these equations can be solved by matrix inversion and time integration; numerical results and approximate formulas are given for thin rings with arbitrary hole radius, namely, the initial slope of the magnetization curve and the field of full penetration in the Bean limit, from which approximate magnetization curves may be constructed. The exact magnetization curves for thin disks and rings are presented in Sec. VII, considering also flux creep and nonconstant critical current density $J_c(B)$ for sinusoidal applied field $H_a(t)$. From these hysteresis loops the nonlinear ac susceptibility $\chi(H_0)=\chi' - \chi''$ for disks and rings is calculated. The main results are summarized in Sec. VIII.

As a particularly useful result, a scaling law is derived in Sec. VI B, which applies to any power-law current-voltage curve $E \propto J^p$ ($E$ is the electric field) and which states that for arbitrary creep exponent $n = \partial \ln E / \partial n J$ the susceptibility $\chi(H_0, \omega)$ depends only on the combination $H_0^{1/n}/\omega$ of the sweep amplitude $H_0$ and sweep frequency $\omega/2\pi$. This scaling connects the well-known facts that in the ohmic ($n=1$) or Bean ($n \to \infty$) limits $\chi$ depends only on $\omega$ or $H_0$, respectively.

II. FEATURES OF THE BEAN MODEL

Several general features follow from the Bean assumption of a static and field-independent $J_c$.

(i) When $H_a$ is cycled between $-H_0$ and $+H_0$ the virgin curve $m(H_a)$ (with $H_a$ increased from zero) determines the full hysteresis loop of $m$. The branches $m_1$ and $m_1$ in de-
creasing and increasing \( H_a \) read

\[
m_1(H_a, H_0) = m(H_0) - 2m \left( \frac{H_0 - H_a}{2} \right),
\]

\[
m_1(H_a, H_0) = - m(H_0) + 2m \left( \frac{H_0 + H_a}{2} \right).
\] (1)

(ii) Only the ratios \( H_a/J_c \) and \( H_0/J_c \) enter in Eq. (1). For general time-dependent \( H_a(t) \) one may write

\[
m[H_a(t)] = J_c t^4 \int [H_a(t)/J_c],
\] (2)

where \( l \) is a characteristic length of the specimen and \( f \) is a dimensionless function, see the examples below.

(iii) As noted recently, \(^3\) for the Bean model the energy dissipated in the superconductor \( U_{\text{diss}} \) follows from \( m(H_0) \) not only for a complete hysteresis cycle, where it is well known to be \( U_{\text{diss}} = \mu_0 \int H_a dH = \mu_0 \int m dH_a \), but for any point on the curve \( m(H_a) \). The dissipated power \( P = dU_{\text{diss}}/dt \) is

\[
P = \int \mathbf{jE} \, d^2r = - \mu_0 \hat{H}_a \left[ m(H_a) - H_a m'(H_a) \right],
\] (3)

where \( m' = dm/dH_a \). With \( \partial H_a/\partial t = \hat{H}_a \partial \hat{H}_a/\partial H_a \) one finds the dissipated energy \( U_{\text{diss}} \) along the vertical curve by integrating \( P = dU_{\text{diss}}/dt = \int \mathbf{jE} \, d^2r \),

\[
U_{\text{diss}} = \mu_0 \int_0^{H_a} H \, dm - \mu_0 \int_0^{H_a} m \, dH
\]

\[
eq \mu_0 H_a m(H_a) - 2 \mu_0 \int_0^{H_a} m(H) \, dH.
\] (4)

Formula (4) means that \( U_{\text{diss}} \) is twice the area between the curve \( m(H) \) and the straight line connecting the origin \( m = H = 0 \) with the point \( m(H_a) \). It appears that the relations (3) and (4) are restricted to the case of constant \( J_c \); they were shown \(^3\) to apply to all the basic Bean examples in longitudinal and transverse geometries listed below, but they do not apply to ohmic conductors. So far, I do not know a general proof.

(iv) In the Bean model the influence of the history of \( H_a(t) \) is partly erased when \( H_a \) is cycled with increasing amplitude. In particular, for periodic \( H_a(t) \) with slowly increasing amplitude, a Bean superconductor of arbitrary geometry ‘remembers’ only the last cycle. It will be shown in Sec. VI that this property approximately holds even when strong creep occurs.

(v) The magnetic moment \( m \) does not depend explicitly on the time \( t \). Therefore, \( m \) does not depend on the sweep rate or, if \( H_a(t) \) is periodic, on the sweep frequency and also not on the shape of \( H_a(t) \) (whether sinusoidal, square, trapezoidal, etc.). This property applies even when \( J_c \) depends on the magnetic field \( B \). It just means that one considers the static response and neglects creep flux.

(vi) The static Bean model follows when a nonlinear current-voltage law \( E(J) \) with a sharp bend at \( J = J_c \) is assumed. A simple such model is to take the limit \( n \rightarrow \infty \) in the power law \( E(J) = E_c (J/J_c)^n J/J \) with \( J = |J| \). This useful and realistic power law with finite creep exponent \( n \) will be used below to investigate creep effects, i.e., effects of finite resistivity.

When the exponent in Eq. (5) is \( n < \infty \), the Bean properties (i) to (v) no longer apply. For \( 1 \ll n < \infty \) the material law (5) leads to flux creep, i.e., a nonlinear, approximately logarithmic relaxation of \( m(t) \) caused by a nonlinear diffusion of magnetic flux with \( J \)-dependent diffusivity \( D(J) = E(J)/\mu_0 J \). With decreasing exponent \( n \) the creep becomes more pronounced. In the limit \( n = 1 \), Eq. (5) describes an ohmic conductor, which exhibits linear diffusion of flux and exponential relaxation of \( m(t) \). Note that for general \( n \) only the combination \( E_c/J_c^2 \) enters in Eq. (5), and thus \( E_c \) (the voltage criterion) and \( J_c \) have no independent meaning. However, in the Bean limit \( n \rightarrow \infty \) only \( J_c \) matters and the choice of \( E_c \) is irrelevant, while in the Ohmic limit \( E_c/J_c = E_c/J_c = \rho \) is the constant resistivity and \( D = \rho \mu_0 \) the usual flux diffusivity in ohmic conductors.

### III. BEAN MAGNETIZATION CURVES FOR BASIC GEOMETRIES

#### A. Slab, cylinder, strip, disk

For later reference I list some basic examples for virgin magnetization curves. A large superconducting slab of width \( 2a \) and area \( A \), and a long cylinder with radius \( a \) and length \( l \), in a parallel field have the magnetic moments \(^1 \)

\[
m_{\text{slab}} = - J_c a^2 A (2h - h^2),
\] (6)

\[
m_{\text{cyl}} = - \pi J_c a^3 (h - h^2 + h^3/3)
\] (7)

for \( 0 \leq h \leq 1 \) with \( h = H_a/H_p \), where \( H_p = J_c a \) is the field of full penetration. For \( H_a \geq H_p \), or \( h \geq 1 \), \( m \) stays constant since the current density has saturated to \( J = J_c \) in the entire sample.

A thin strip \(^4\)–\(^7\) of width \( 2a \) and length \( l \) \( \gg a \) and a circular disk \(^8\)–\(^10\) of radius \( a \), both with thickness \( d \ll a \), in a transverse field \( B_a \) have the magnetic moments \( m_{\text{strip}}(H_a) \) and \( m_{\text{disk}}(H_a) \) (8)–(9)

\[
m_{\text{strip}}(H_a) = - J_c da^2 \tan hh,
\] (8)

\[
m_{\text{disk}}(H_a) = - J_c da^2 \frac{2}{3} \left[ \cos^{-1} \left( \frac{1}{\cosh h} \right) - \frac{\sinh |h|}{\cosh^2 h} \right]
\] (9)

for \( 0 \leq h < \infty \) with \( h = H_a/H_c \), where \( H_c = J_c d/\pi \) for the strip and \( H_c = J_c d/2 \) for the disk. The two curves (8) and (9) differ by less than 0.012 if normalized to the same initial slope \( m(0) \) and same saturation value \( m(\infty) \), cf. Fig. 1, where we plot the universal functions \( f(x) = m_{\text{strip}}/(-J_c da^2 l) \) with \( x = \pi H_a/J_c d \) and \( f(x) = m_{\text{disk}}/(-\pi J_c da^2 l/3) \) with \( x = 8H_a/J_c d \), which both exhibit \( f'(0) = f(\infty) = 1 \) and almost coincide. The computed normalized Bean magnetic moment \( m_{\text{square}}(H_a) \) for a thin superconducting square \(^11\)–\(^12\) differs from \( m_{\text{disk}}(H_a) \) by less than 0.002.

In the considered limit \( d/a \rightarrow 0 \), the transverse moments \( m_{\text{strip}}(H_a) \) and \( m_{\text{disk}}(H_a) \) saturate only at infinitely large field \( H_a \). With finite thickness \( d \) the magnetic saturation is reached when the flux and current front, which is positioned
FIG. 1. Bean magnetization curves \( m(H_a) \) for a thin strip [Eq. (8)] and disk [Eq. (9)], and for rings with radius ratios \( a_1/a = 0.4, 0.6, 0.8, 0.9, \) and 0.95 in perpendicular field \( H_a \). The curves are normalized to unity initial slope and unity saturation value by plotting \( m(H_a)/m_{\text{sat}} \) versus \( H_a m'(0)/m_{\text{sat}} \). Also shown is \( m(H_a)/m_{\text{sat}} \) for long slabs [Eq. (6)] and cylinders [Eq. (7)] in parallel field, plotted versus \( H_a/H_p \) to avoid crossing of curves. The inset shows \( m(H_a) \) for these rings and the disk in constant units \( m_{\text{sat}} = \pi J_c d a^2/3 \) for \( m \) and \( J_c d \) for \( H_a \), which show how the magnetic saturation is reached earlier with increasing hole radius \( a_1 \). The ring data were computed using a creep exponent \( n = 101 \).

at \( x_p = r_p = a/\cosh h \), has penetrated to a distance \( \approx d/2 \) from the center \( x = 0 \) or \( r = 0 \), i.e., to \( x_p = r_p = d/2 \). More precisely, one finds from analytical calculations\(^{13} \) the penetration fields\(^{14} \) for \( d \ll a \),

\[
H_p^{\text{strip}} = J_c d \frac{1 + \ln(2a/d)}{2}, \quad (10)
\]

\[
H_p^{\text{disk}} = J_c d \frac{2 + \ln(2a/d)}{2}. \quad (11)
\]

The prefactors in Eqs. (10) and (11) are the above defined critical fields of strips and disks, \( H_c = J_c d/\pi \) and \( H_c = J_c d/2 \). Above the penetration fields (10) or (11) the magnetic moment saturates to the values \( m_{\text{sat}}^{\text{strip}} = J_c d a^2 \pi/3 \) and \( m_{\text{sat}}^{\text{disk}} = J_c d a^3 \pi/3 \).

**B. Narrow ring**

The cutoff problem for \( H_a \) does not arise if the central part of the strip or disk is removed. The magnetic moment of the resulting double strip (with connected ends) or ring then saturates naturally, with the penetration field \( H_p \) and the saturated magnetic moment depending on the width or radius of the cut-out slit or hole. For the thin double strip the Bean magnetic moment \( m(H_a) \) in principle may be calculated by conformal mapping as suggested in Ref. 6. For both the double strip and the ring the numerical method of Refs. 15–20 may be used to compute the flux penetration and the magnetic moment, cf. Sec. VI.

First I discuss the limit of a thin narrow ring with width \( w \) much smaller than the mean radius \( R \), \( a_1 = R - w/2 \leq r \leq R + w/2 = a \). The virgin magnetization curve of such a ring is composed of two straight lines,

\[
m_{\text{ring}}(H_a) = -(H_a/H_p^{\text{ring}}) m_{\text{sat}}, \quad H_a \leq H_p^{\text{ring}},
\]

\[
m_{\text{ring}}(H_a) = -m_{\text{sat}}, \quad H_a \geq H_p^{\text{ring}}.
\]

(12)

This is so since the screening supercurrent in the ring is limited to a maximum value \( I = J_c d w \). As long as the current \( I \) induced in the ring by the applied field \( H_a \) is \( |I| < I_c \), no magnetic flux can penetrate through the ring into the hole of the ring. When \( |I| = I_c \) is reached, the ring becomes transparent to magnetic flux. Therefore, when the applied field is increased further, flux lines will move through the ring as described in Refs. 6 and 7, which treat the superconducting strip with transport current in an applied field, see also the flux profiles in Fig. 10 below. The flux lines transport magnetic flux into the ring hole until the screening current has decreased again to the value \( I_c \). The magnetic moment \( m = \pi R^2 I \) of the ring thus saturates to the value

\[
|m| \leq m_{\text{sat}} = \pi R^2 I_c.
\]

(13)

The applied field value \( H_p \) at which this saturation is reached follows from the inductivity \( L \) of the flat ring,

\[
L = \mu_o R \left( \frac{8 R}{w} - c \right) \quad (14)
\]

where \( c = 1/2 \). For comparison, the inductivity of a ring made of a wire with diameter \( w \ll R \) is also given by formula (14) but with \( c = 2 \). Our computation in Sec. VII finds that for a flat superconducting ring one actually has \( c = 0.614 \) in formula (14). This is so because the result \( c = 1/2 \) assumes uniform current density across the ring, while the real current distribution is peaked at the edges of the ring.\(^{5,7} \) In the limit \( w \ll R \) the sheet current \( J_s \) in a superconducting ring behaves as in a straight thin strip, \( J_s(r) = (I/\pi)(w^2/4 - r^2)^{-1/2} \) with \( x = r = R \). Therefore, when flux penetrates the conductor of a flat ring, the constant decreases from \( c = 0.614 = 2 - \ln 4 \) at \( H_a = 0 \) to \( c = 1/2 \) (or \( c = 0.307 = 1 - \ln 2 \), see Sec. VII B) at \( H_a \approx H_p^{\text{ring}} \). For simplicity one may put \( c = 1/2 \) in the next three formulas.

The magnetic flux generated in the ring hole by a ring current \( I \) is \( \phi = LI \). As long as \( I < I_c \), one has ideal screening, thus \( \phi = -\pi R^2 \mu_o H_a \). Equating these two fluxes one obtains for the flat ring

\[
I = \phi/L = -\frac{\pi R}{\ln(8R/w) - c} H_a. \quad (15)
\]

At \( H_a = H_p \) one has reached \( |I| = I_c = J_c d w \), thus

\[
H_p^{\text{ring}} = \frac{\ln(8R/w) - c}{\pi R} I_c. \quad (16)
\]

The slope of \( m(H_a) \) for \( H_a < H_p \) is therefore

\[
m_{\text{ring}}'(0) = -m_{\text{sat}} H_p^{\text{ring}} = \frac{\pi^2 R^3}{\ln(8R/w) - c}. \quad (17)
\]

Comparing this with the ideal diamagnetic moment of the disk obtained from Eq. (9),

\[
m_{\text{disk}}'(0) = -8a^3/3, \quad (18)
\]
one finds the remarkable result that a ring has a slightly smaller ideal diamagnetic moment than a disk of the same outer radius \( a = R + w/2 \gg w \), although the flux expelled from the ring and disk is the same, \( \pi a^2 \mu_0 H_0 \).

The initial slope \( m'(0) \) and the penetration field \( H_p \) are computed below in Sec. VII for superconducting rings with arbitrary inner radius \( 0 \leq a_1 < a \). The results, depicted in Figs. 2 and 3, show that the expressions (16) with \( c = 0.307 = 1 - \ln 2 \) and (17) with \( c = 0.614 = 2 - \ln 4 \) are excellent approximations with a relative deviation of less than 2% for \( a_1/a > 0.8 \). Figure 1 shows that the magnetization curves of wide rings are no longer straight lines but are slightly curved and merge with the curve for the disk when \( a_1 \to 0 \).

**IV. SUSCEPTIBILITY OF NARROW RINGS**

With the prescription (1) the virgin magnetization curve (12) yields hysteresis loops in the shape of parallelograms. From these loops one obtains the complex susceptibility \( \chi = \chi' - i \chi'' \) of a superconducting narrow ring with constant \( J_c \). For cycled magnetic field \( H_s(t) = H_0 \sin(\omega t) \) one may define

\[
\chi(H_0) = \frac{\omega}{\pi H_0} \int_0^{2\pi} m(t) e^{-i\omega t} dt .
\]

Note that this nonlinear susceptibility does not depend on the frequency \( \omega/2\pi \) if \( m(t) \) is taken from the Bean model, but it depends on the sweep amplitude \( H_0 \). In contrast, the linear susceptibility of an ohmic conductor or of any conductor with linear complex resistivity (e.g., a superconductor above the depinning line) depends on \( \omega \) but not on \( H_0 \). One may also consider the higher harmonic responses \( \chi_n \), \( n = 2, 3, 4, \ldots \), by replacing in Eq. (19) \( e^{-i\omega t} \) by \( e^{-i\omega nt} \). Here I shall consider the fundamental response \( \chi_1(H_0) \) and the strongest harmonics \( \chi_3(H_0) \). The susceptibilities \( \chi_n(H_0) \) for thin disks are given by Clem and Sanchez.10

From Eqs. (1), (12), and (19) one obtains the susceptibility of the ring normalized to the initial value \( \chi(0) = -1 \), i.e., to \( \chi - \chi/m'(0) \),

\[
\chi'(h) = -1, \quad \chi''(h) = 0, \quad h \equiv 1,
\]

\[
\chi'(h) = -\frac{1}{2} - \frac{1}{\pi} \arcsin \frac{1}{\sqrt{1 - s^2}} - \frac{1}{\pi} \arcsin \frac{1 - s^2}{\pi}, \quad h \equiv 1,
\]

\[
\chi''(h) = \frac{4}{\pi} \frac{h - 1}{h^2} = \frac{1 - s^2}{\pi}, \quad h \equiv 1,
\]

with \( h = H_0/H_p \) and \( s = 2h - 1 \), see Fig. 4 (top).

Interestingly, with the ring susceptibility (20) the polar plot \( \chi''(\chi') \) versus \( \chi' \) with \( h \) as a parameter is symmetric, i.e., \( \chi''(\chi') \) yields the same curve as \( \chi''(-1 - \chi') \). Figure 4 (bottom) shows these polar plots for the ring and for the disk and strip. The maximum of the dissipative part of the ring susceptibility \( \chi''_{\max} = 1/\pi = 0.318 \) occurs at \( h = 2 \) (at \( s = 0 \)). For large amplitudes \( h = H_0/H_p \gg 1 \), one has \( \chi'(h) \approx -1.69/4^{3/2} \) and \( \chi''(h) \approx 4\pi h \).

The \( \chi(H_0) \) for disks and strips differ by less than 0.010 \( \chi''(\chi') \) or 0.0043 \( \chi''(\chi') \), and \( \chi(H_0) \) for thin squares and disks differ by less than 0.001. The maximum \( \chi''_{\max} = 0.240 \) (0.236 466) for the disk (strip) occurs at \( x = 8H_0/\pi J d = 2.4738 \) or \( x = \pi H_0/\pi J d = 2.4642 \); for these field units see the discussion below Eq. (9). For large amplitudes \( h \gg 1 \) one has \( \chi'' = 1.910\chi_{3/2} / (1.930\chi_{3/2}^3 \) for disks (strips) and \( \chi'' = 4\pi x \) for both. This means that the large amplitude behavior of \( \chi(H_0) \) is very similar for rings, disks, and strips, if the amplitude \( H_0 \) is expressed in the reduced units of Fig. 1, which yield unity initial slope and unity saturation of \( m(H_0) \).

A further remarkable property of the ac response of a narrow ring within the Bean model is that the absolute value of \( \chi_3 \) is related to the dissipative part of \( \chi \). Eq. (20), by \( |\chi_3(H_0)| = |\chi''(H_0)\). This exact coincidence was also noted by Ishida and Mazaki,24 who proved it theoretically and found nice agreement with experiments. For higher harmonics,
FIG. 4. Top: Real and imaginary parts of the complex susceptibility \( \chi(H_0) = \chi' + i \chi'' \) of thin strips, disks, and rings with various radius ratios \( a_1/a \) plotted versus the amplitude \( H_0 \) of the ac field in the Bean model with constant \( J_c \). The unit of \( H_0 \) is \( J_c d/\pi \) for the strip and \( \pi J_c d/8 \) for the disk. For rings \( H_0 \) is in units of a penetration field \( H_p = H_p^\ast \), which gives best fit to the sharp rise in the ideal ring \( \chi' \). \( H_p/H_p^\ast = 0.126 \) (0.210) for \( a_1/a = 0.9 \) (0.8), while from Table I one has \( H_p/J_c d = 0.136 \) (0.239). For \( a_1/a \to 1 \) one finds \( H_p/H_p^\ast \to 1 \). Bottom: Polar plots of the same \( \chi = \chi' - i \chi'' \). For the narrow (not necessarily thin) ring \( \chi' \) and \( \chi'' \) [Eq. (20)] are discontinuous at \( H_0 = H_p \) and the polar plot is symmetric.

ics \( n = 5, 7, \ldots \) a similar relationship does not apply. We will see in Sec. VIII below that this property is lost when flux creep is allowed.

For comparison with the Bean limit \( n \to \infty \) shown in Fig. 4, the complex susceptibility for disks and various rings is depicted in Fig. 5 for finite creep exponents \( n = 3, 5, 11, 51, \) and 201. These curves were computed by the method of Sec. VI using the current-voltage power law (5) and \( \omega = 2E_c/(\mu_0 J_c d a) \). Note that with increasing hole radius \( a_1 \), a bend develops in the curves \( \chi'(H_0) \), \( \chi''(H_0) \), and \( \chi''(-\chi') \) at \( H_0 = H_p \). This bend is sharpest in the Bean limit \( n \to \infty \) but it dissappears for strong creep, \( n \ll 5 \). The tiny wiggles of \( \chi' \) and \( \chi'' \) at \( H_0 = H_p \) are artifacts caused by the spatial grid used for the computation, but the bends near \( H_p \) are real. At small amplitudes \( H_0 = H_p \), \( \chi' \) and \( \chi'' \) for rings with finite width deviate considerably from the narrow ring Bean limit (\( \chi' = -1, \chi'' = 0 \)) even for large \( n \).

**V. EFFECTS OFFINITE RESISTIVITY**

Within a continuum description the phenomenon of flux creep is caused by the finite resistivity of the superconductor or of any conductor with a nonlinear current-voltage curve \( E(J) \) with positive curvature. Various \( E(J) \) laws were suggested based on microscopic models. For example, the collective creep\( ^{28} \) and vortex glass pictures\( ^{29} \) yield a current-density-dependent activation energy for thermally activated depinning,

\[
U(J) = U_0[(J_c/J)^n - 1]/\alpha
\]

with \( \alpha > 0. \)

Inserting this into the Arrhenius law

\[
E(J) = E_c \exp[-U(J)/kT]
\]

one obtains a highly nonlinear resistivity \( \rho(J) = E(J)/J \). In the limit \( \alpha \ll 1 \), Eq. (21) becomes \( U(J) \approx U_0 \ln(J_c/J) \), yielding with Eq. (22)

\[
E(J) = E_c \exp[-U(J)/kT]
\]

This is the power law, Eq. (5), with \( n = U_0/kT \). Such a power law, or a logarithmic \( U(J) \), or a small glass exponent \( \alpha \ll 1 \), is often observed in experiments.

The power law (5) is very convenient for analytical calculations\( ^{31–36} \) and for computations.\( ^{31–45} \) In creep experiments where the ramping of \( H_a(t) \) is stopped at time \( t = 0 \) one obtains after some transient time \( \tau \) the universal relaxation\( ^{33–36} \)

\[
E \propto (\tau/t)^{(n-1)/3} \tau/t \quad \text{and} \quad m(t) \propto J \propto (\tau/t)^{(n-1)/3} = 1 - [1/(n-1)] \ln(\tau/t) \quad \text{if} \quad n \ll 1.
\]

In Eq. (5) general dependences \( J_c = J_c(B) \) and \( n = n(B) \) are allowed. Of course, further dependences on temperature and on any other spatially constant parameter are also allowed, as in the original Bean model. Static \( (n \to \infty) \) extensions of the Bean model to \( J_c = J_c(B) \) have often been considered,\( ^{2,37} \) but also dynamic computations with creep, using Eq. (5) with \( n < \infty \), present no difficulty.\( ^{18,19} \) Superconductors with inhomogeneous pinning, produced, e.g., by partial irradiation, are easily modeled by using a space dependent \( J_c(\mathbf{r}) \).\( ^{19,20} \)

One can easily calculate the fully penetrated state which is reached after sufficiently long time \( t \to \infty \) during constant ramp rate \( H_a \). In this stationary state the current has saturated, thus the current-caused magnetic field does not change any more, and the electric field \( E_{\text{sat}} \) follows from

\[
E_{\text{sat}} = E_c \hat{\phi} \mathbf{B}_a \frac{r}{2},
\]

where \( \hat{\phi} \) is the azimuthal unit vector and \( \mathbf{B}_a = \mu_0 H_a \). From the current-voltage law (5), which inverted reads \( J = J_c(E/E_c)^{1/n} \), one obtains with Eq. (23)

\[
J_{\text{sat}} = \frac{\hat{\phi} \mathbf{B}_a}{2E_c} \left( \frac{r}{a} \right)^{1/n} \frac{1}{a}. \quad (24)
\]

The saturated magnetic moment of a ring with outer radius \( a \), inner radius \( a_1 = a - w \), and thickness \( d \) is thus

\[
|m| \leq m_{\text{sat}} = d \pi \int_{a_1}^{a} r^2 J(r) \, dr,
\]

where

\[
H_a(t) = \begin{cases} \frac{t}{\tau} & \text{for } 0 \leq t \leq \tau, \\ 1 & \text{for } t > \tau. \end{cases}
\]
FIG. 5. (a) Top: Real and imaginary parts of the nonlinear complex susceptibility $\chi(H_0) = \chi' - i\chi''$ of a thin disk plotted versus the ac amplitude $H_0$ for various creep exponents $n = 3, 5, 11, 51,$ and 201 in $E = E_c(J/J_c)^n$ with constant $J_c$. $H_0$ is in units $J_c d$. Bottom: Polar plots of the same $\chi' - i\chi''$. The dashed line shows the Bean model $n \to \infty$ for the disk. (b) Same as (a) but for a wide ring with radius ratio $a_1/a = 0.4$. (c) Ring with $a_1/a = 0.6$. (d) Ring with $a_1/a = 0.9$. The dashed curves show $\chi' - i\chi''$ for the narrow Bean ring with $a_1/a \to 1$ and for the disk; for the field units see the caption of Fig. 4.
In the Bean limit $n \to \infty$ the first factor in Eq. (25) reduces to $\pi r^2$ and the last factor becomes unity, i.e., the ramp rate $B_a$ and voltage criterion $E_c$ do not enter. One then obtains the well-known exact Bean result

$$m_{\text{sat}} = \frac{\pi n}{3n+1} J_c a^3 d \left[ 1 - \left( \frac{a_1}{a} \right)^{3+(1/n)} \left( \frac{B_a a}{2E_c} \right)^{1/n} \right].$$

The magnetic saturation is reached quite fast when the ramp rate is constant. We will see in Sec. VII C that when $J_s(t)\to J_s(t)$ during increased applied field $B_a(t) = B_0 \sin \omega t$ was ascribed to flux creep, which dominates when the ramp rate $B_a(t) = B_0 \omega \cos \omega t$ becomes sufficiently small. However, a more appropriate quantiative interpretation\textsuperscript{38} ascribes this decrease of $|m|$ to the decreasing ramp rate $[B_a]$. Namely, when the current density $J$ has nearly saturated, the electric field decreases like the ramp rate, $|E| \approx (r/2)B_a$, thus also $|J| = J_s|E|/E_c^{1/n}$ decreases, and $m = m(B_a)$ is given by the ramp-rate-dependent saturation value, Eq. (25).

VI. EXACT FORMULATION FOR RINGS

A. Equation of motion for $J$

The equation of motion for the current density $J(r,t)$ in disks and rings in an axial magnetic field $B_a(t)$ is obtained by generalizing the expressions for disks of finite thickness\textsuperscript{14} or for thin disks\textsuperscript{17-20} with $d \ll a$. For thick rings with $a_1 \leq r \leq a$, $-d/2 \leq z \leq d/2$ this equation reads

$$J(r,t) = \mu_0^{-1} \int_{a_1}^{a} \int_{-d/2}^{d/2} dz' \int_{-d/2}^{d/2} dz'' Q(r, r')^{-1} f(r', t'),$$

$$f(r,t) = J(r,t) - \frac{E_c}{J_c} - \frac{r}{2} B_a(t).$$

Here $r = (r,z), r = (x^2+y^2)^{1/2}$, $f(r,t)$ is the electric field which drives the current (the emf induced by the varying applied field, minus the voltage drop), and $Q(r,r')^{-1}$ is the inverse of the (negative) integral kernel

$$Q(r,r') = c(q, r', z - z') + q(r, r', z + z'),$$

$$q(r, \xi) = \int_{0}^{\pi} \frac{d \phi}{2 \pi} \frac{r' \cos \phi}{(r^2 + r'^2 + 2rr' \cos \phi)^{1/2}}.$$
method is thus very effective both for numerical computations and for analytical calculations, see Refs. 14,15,17, 34,36,41.

After the currents have been calculated from Eqs. (28) or (30), the magnetic field inside and outside the specimen can be obtained by the Biot-Savart law. Writing \( \mathbf{B} = \nabla \times \mathbf{A} + \mathbf{B}_a \) with \( \mathbf{A} = \hat{\mathbf{r}} A(r,z) \), one gets the components \( \mathbf{B}_z = \partial A/\partial r + A/r + \mathbf{B}_a \) and \( \mathbf{B}_r = -\partial A/\partial z \) with

\[
A(r) = -\mu_0 \int_{a_1}^{a} dr' \int_{0}^{d_2} dz' Q(r,r')J(r') \quad . (33)
\]

For a thin disk or ring one has at the surfaces \( z=\pm d/2 \leq a \) the field components \( \mathbf{B}_z = \partial A(r,0)/\partial r + A(r,0)/r + \mathbf{B}_a \) and \( \mathbf{B}_r = \pm J_s(r)/2 \) (since the jump of the parallel component \( \mathbf{B}_r \) equals the sheet current \( J_s \) with

\[
A(r) = -\mu_0 \int_{a_1}^{a} dr' Q(r,r')J_s(r') \quad . (34)
\]

B. Scaling of frequency and amplitude

When a power law is assumed for the current-voltage curve, \( E \propto J^n \) or \( p = E/J \propto J^{n-1} \) with \( n = 1 \), cf. Eqs. (5) and (22), then the equations of motion (28) and (30) for the currents in thick and thin disks or rings (and in all other geometries) possess a remarkable scaling property. Namely, when one changes the time units by a constant factor of \( \hat{\mathbf{r}} = t/C \) and the current and field units by a factor of \( C^{1/2} \), where \( n = 1 \), then these equations are invariant, i.e., the same solutions result for these scaled quantities. Explicitly, if the equation for the current density \( J(r,t) \) is expressed in terms of \( \hat{\mathbf{r}} = t/C \) then the new functions

\[
\hat{\mathbf{B}}(r,\hat{\mathbf{r}}) = \mathbf{B}(r,t)C^{1/2} \quad , \quad \hat{\mathbf{B}}_a(\hat{\mathbf{r}}) = B_a(t)C^{1/2} \quad . (35)
\]

satisfy the same equation of motion. The resulting magnetic field \( \hat{\mathbf{B}} \) and magnetic moment \( \hat{m} \) scale by the same factor as \( J \).

\[
\hat{\mathbf{B}}(r,\hat{\mathbf{r}}) = \hat{\mathbf{B}}(r,t)C^{1/2} \quad , \quad \hat{m}(\hat{\mathbf{r}}) = m(r)C^{1/2} \quad . (36)
\]

In particular, for periodic \( B_a(t) = B_0 \sin(\omega t) \), the resulting complex susceptibility \( \chi(B_0,\omega) \) normalized to \( \chi(0,\omega) = \chi(B_0,\infty) = -1 \), i.e., to the ideal diamagnetic limit occurring at zero amplitude or infinite frequency, remains unchanged if one increases, e.g., \( \omega \) by a factor of 10 and \( B_0 \) by a factor of \( 10^{1/2} \). The resulting magnetic moment is then larger by the same factor of \( 10^{1/2} \) as is also the applied field \( B_a \). Therefore, the shape of the hysteretic magnetization curve \( M(B_0) \) remains unchanged and also the polar plots \( \chi''(\chi') = \chi'' - i\chi' \).

For periodic \( B_a(t) \) this scaling law states that the normalized \( \chi(B_0,\omega) \) and the shape of the hysteresis loop \( m(B_0) \), depend only on the combination \( B_0/\omega^{1/2} \) or on \( \omega/B_0^{1/2} \). This general statement connects the two limiting cases of ohmic conductors, exhibiting \( \sigma = 0 \) and thus \( \chi = \chi(\omega) \), and of Bean superconductors, exhibiting \( \sigma \to \infty \) and thus \( \chi = \chi(B_0) \).

C. Computational method

From Eqs. (30) to (34) one may compute the sheet current, the irreversible magnetization curves, and the magnetic field in and around thin disks and rings with material laws \( \mathbf{B} = \mu_0 \mathbf{H} \) and \( \mathbf{E} = E_J \mathbf{J}/|\mathbf{J}| \) in a perpendicular \( B_a(t) \). Since these equations are one dimensional in space, the numerical program is very fast and accurate if the following tricks are considered

(i) For the spatial integration a nonequidistant grid \( r_i \) may be chosen which has vanishing spacing (or vanishing weights \( w_i = \Delta r_i/|dr_i| \) at the inner and outer edges \( r = a_1 \) and \( r = a_2 \), e.g., \( r_i = a_1 + (3/2)u_i(a-a_1)/(a-a_i) \), \( w_i = 6u_i(1 - u_i)(a-a_2)/N \), with \( u_i = (i-\frac{1}{2})/N \), \( i = 1,2, \ldots, N \). For the computation of \( J(r) \) only grid points inside the ring are required, \( a_1 \leq r_i \leq a_2 \). The subsequent computation of \( B_a(r,j) \), and \( B_a(r,j) \) may also use grid points in the ring hole and outside the ring in the plane \( r = 0 \). In principle, \( B_a(r,z) \), and \( B_a(r,z) \) may be calculated also outside the disk plane by using a two-dimensional grid \( r_j = (r,z) \), and the kernel \( Q(r,r') \) (29).

(ii) Integrals over any function defined in the interval \( a_1 \leq r \leq a_2 \) may be evaluated as a weighted sum, e.g., writing \( f_i = f(r_i) \), and \( F_i = F(r_i) \),

\[
\int_{a_1}^{a} Q(r_i,r')f(r') \, dr' \approx \sum_{j=1}^{N} Q_{ij} w_j f_j = F_i . (37)
\]

(iii) When tabulated on the grid \( r_i \) the kernel \( Q(r,r') \) becomes a two-dimensional matrix \( Q_{ij} = Q(r_i,r_j) \). The inverse kernel \( Q(r,r')^{-1} \) is the inverse of this matrix if equidistant grid points (constant weights \( w_i \) are used. For a general grid one has to invert this matrix times the weights \( w_i \) (\( Q^{-1} \) is \( Q_{ij} w_j \) as can be seen by inverting Eq. (37).

\[
\sum_{j=1}^{N} (Q_{ij} w_j)^{-1} F_j . (38)
\]

(iv) The time step \( dt \) used for the time integration of \( J \) should be chosen as \( dt = c_1/[\max(p + c_2)] \), where \( p = E/J = (E_J/E_c) |J|J_c^{n-1} \) and \( c_1 \), and \( c_2 \) are constants. A finite \( c_2 \) is required to limit the time step at the moment when the ramp rate \( B_a(t) \) changes sign.

D. Current and field profiles

Figures 6 to 9 show the sheet current \( J_s(r) \) and the corresponding magnetic field profiles \( B(r) = B_a(r) \) in the specimen plane \( z = 0 \), computed for the thin disk \( (a_1 = 0) \) and for a wide (\( a_1/a = 0.4 \)) and a narrow (\( a_1/a = 0.8 \)) ring. Shown are the Bean limit (\( n = 101 \), Fig. 6) with \( J_c = \text{const in increasing (Fig. 6) and decreasing (Fig. 9) field } B_a \). Also shown is an example with field-dependent critical current \( J_c \) (Kim model, Fig. 7, \( J_s \), and \( B/J_c \) in units \( J_c d \) and one case with strong creep \( n = 9 \) (Fig. 8). Note that in Fig. 9 in decreasing \( B_a(t) \) the \( J_s \) profiles (but not the \( B \) profiles) nearly coincide with the \( J_s \) profiles of Fig. 6 in increasing \( B_a(t) \), multiplied by \(-2 \) and then shifted up by \(+ J_c \), as it should be in the Bean model.\(^{1,2,58}\) see also Eq. [58].
For stronger creep (Fig. 8) the corners of the $J_s$ profiles become rounded but not the $B$ profiles.

VII. MAGNETIZATION CURVES

A. Initial slope

The initial slope of the magnetization curve $m'_{\text{ring}}(0)$ with the limits (17) $(a_1/a-1)$ and (18) $(a_1/a-0)$, for arbitrary radius ratio $a_1/a$ may be directly obtained from the inverted matrix $Q(r,r')^{-1}$ in the following way. Inverting Eq. (34) with the applied vector potential $A=-rB_a/2$ inserted, one obtains the sheet current which ideally screens the applied field $B_a=\mu_0 H_a$ from the ring (more precisely, it shields the perpendicular flux from the hole of the ring and the perpendicular field from the ring material),

$$J_{s,\text{ideal}}(r) = -\frac{1}{2} H_a \int_{a_1}^a dr' Q(r,r')^{-1} r'. \quad (39)$$

FIG. 6. Profiles of the sheet current $J_s(r)$ (top) and the perpendicular magnetic field $B_B=B_s(r)$ in the specimen plane (bottom) in a disk and in rings with radius ratios $a_1/a=0.4$ and $0.8$ during flux penetration in an increasing applied field $H_a(t)$. The ten depicted curves are for $H_a/H_p=0.1, 0.2, \ldots, 1$ with the penetration field $H_p=2J_d d$ for the disk chosen arbitrarily and $H_p=0.6699 J_d d (0.2389 J_d d)$ for the two rings taken from Table I. Bean model with $J_c=\text{const}$, $n=101$, $J_c$ in units $J_d d$ and $B$ in units $\mu_0 J_d d$. A grid of 40 nonequidistant points $a_1<r_i<a$ was used.

Inserting this into Eq. (32) one obtains the magnetic moment of ideal superconducting thin rings and disks divided by $H_a$,

$$m_{\text{ring}}'(0) = -\frac{\pi}{2} \int_{a_1}^a dr \int_{a_1}^r dr' r^2 Q(r,r')^{-1} r'. \quad (40)$$

In terms of a discrete grid $r_i$ with weights $w_i$ one may write this explicitly as

$$m_{\text{ring}}'(0) = -\frac{\pi}{2} \sum_{i,j} r_i^2 w_i (Q_{ij} w_j)^{-1} r_j. \quad (41)$$

Evaluating this double sum as a function of the radius ratio $a_1/a$ of the ring one obtains the limits (18) for full disks and (17) for narrow rings, and the values between these limits. Denoting the ratio of the inner and outer radius by $x=a_1/a$ one gets the explicit formulas,

$$m_{\text{ring}}'(0) = \begin{cases} \frac{8}{3} \left(1 - \frac{1}{3} x^5 - 0.21 x^{10}\right), & x \leq 0.95 \\ \frac{\pi^2}{8} \left(\frac{1+x^3}{1-x} \right) - 0.614, & x \geq 0.9. \end{cases} \quad (42)$$

The expressions (42) describe the correct limits $x \leq 1$ and $1-x \ll 1$; their relative error is $<0.001$ for $x<0.9$ and $<0.01 (0.003, 0.001)$ for $x>0.9 (0.95, 0.975)$, cf. Fig. 2 and...
The excellent fit over many decades suggests that the factor $1 - 0.111 \times 5$ in Eq. (42) presents the exact limiting behavior for $x \approx 1$, which should be derivable analytically. Also it seems likely that the exact value of the constant $0.614$ should be $2 - \ln 4 = 0.614$. A general interpolation formula with relative error $< 2\%$ for all radius ratios $0 < x = a_1/a < 1$ is

$$
m_{\text{ring}}(0) = \frac{2.064 + 5.351x + 2.456x^{3.5}}{\ln \left[ 16 \left( \frac{1+x}{1-x} \right) \right]} - 2.
$$

(43)

Note that the initial slope $m'(0)$ does not depend on the creep exponent $n$ as long as $n > 1$: The nonlinearity of $E \propto J''$ guarantees that in the limit of small $H_a$, and thus small $J$, any nonlinear conductor is an ideal conductor which screens perfectly.

### B. Field of full penetration

The computation of the field of full penetration $H_{p}^{\text{mag}}$ for rings with arbitrary radius ratio $0 < x = a_1/a < 1$ and finite creep exponent $n$ requires time integration of Eq. (30). Using constant ramp rate $B_a$ and a large creep exponent $n \gg 1$ one is still left with some arbitrariness in defining $H_p$ since for finite $n$ the magnetic saturation is reached only gradually, see Eq. (27). One possible definition of $H_p$ by monitoring the vanishing spatial variation of the current density $J(r,t)$ was used in Ref. 14. Here I use a slightly different definition in terms of the magnetization curve $m(H_a)$. The computed function $f(H_a) = \ln(\partial m/\partial H_a)$ performs a rather sharp bend from a nearly horizontal line at $H_a = H_p$, where $\partial m/\partial H_a = m'(0)$, to a steeply decreasing straight line at $H_a > H_p$, which reflects the exponentially decreasing slope of $m(H_a)$, Eq. (27). Here I define $H_p$ as the $H_a$ value at this bend.

To obtain the Bean limit $n \to \infty$ by this method I used $n = 200$ and $B_a = 2E_c/a$, or $B_a = 1$ in units $a = 2E_c$, $J = 1$. The penetration field $H_{p}^{\text{mag}}$ computed in this way is in good agreement (relative error $< 1\%$) with the exact value obtained directly from the integral kernel $Q(r,r')$ in the following way.

Looking at the profiles of the perpendicular magnetic field $B_z(r)$ in the ring plane $z = 0$ during increase of the applied field $H_a$, Fig. 6, one notes that magnetic flux enters the ring material from the outer and inner edges. As long as the penetrating flux fronts (which in this transverse geometry exhibit vertical slopes $\propto n$) have not yet met, there is a field-free zone with $B_z = 0$ between these two fronts. When $H_a$ has reached the value $H_{p}^{\text{mag}}$, the flux fronts meet at $r = r_p$ and the sheet current $J_s$ saturates to the value $J_s$. At this moment, there is still no flux in the circular zone $r < r_p$, thus,

$$
\int_0^{r_p} 2\pi r B_z dr = 2\pi r_p A(r) + \pi r_p^2 B_p = 0.
$$

(44)
Here I used \( rB_z = \partial (ra) / \partial r + ra \) and \( B_a = B_p = \mu_0 H_p \). The current-caused vector potential \( A(r) \) in the plane \( z = 0 \) is given by Eq. (34) with \( J_s (r) = J_c \), and the field of full penetration \( H_p \). \( a \) is obtained if in Eq. (16) the constant \( c \), which in the discussion following Eq. (14) was chosen to be \( 1/2 \), is chosen as \( c = 0.307 \approx 1 - \ln 2 \). The reason for this different \( c \) value is the ambiguity of the definition of the self-inductivity \( L \) of a ring.
FIG. 10. Theoretical magnetization curves \( m(H_a) \) of a thin disk (radius \( a \)) with \( J_c = \text{const} \) (top) and \( J_c = J_c(B) = J_c(0)/0.8 + |B| \) (bottom). \( B \) is in units \( \mu_0 J_c d m \) in units \( J_c da \). Solid lines: Weak creep, \( n = 51 \) (Bean and Kim models). Dashed lines: Strong creep, \( n = 5 \) in \( E(J,B) = E_r[J/|J_c(B)|]^{n} \). The curves for \( n = 11 \) would lie in the middle between the curves \( n = 5 \) and \( n = 51 \). The applied field is cycled sinusoidally, \( H_a(t) = H_0 \sin \omega t \), starting at time \( t = 0 \). The stationary hysteresis loop is reached before the first cycle is completed.

Such computed magnetization curves are depicted in Figs. 10 and 11 for the Bean limit \((n = 51)\) and for strong creep \((n = 5)\) for the thin disk and a ring with radius ratio \( a_1/a = 0.6 \). The upper plots show the Bean model \( J_c = \text{const} \), and the lower plots the Kim model \( J_c(B) = J_c(0)/0.8 + |B| \) with \( B \) in units \( \mu_0 J_c d \). The applied field is cycled sinusoidally, \( H_a(t) = H_0 \sin \omega t \) with \( \omega = 2E_r/(\mu_0 J_c da) \). Shown are the virgin curves and hysteresis loops for amplitudes \( H_0/J_c d = 0.2, 0.6, 1, 1.5, \) and 2. Note that even this wide ring exhibits almost parallelogram-shaped hysteresis loops, which are typical for a narrow ring, cf. Eqs. (1) and (2).

With decreasing creep exponent \( n \) the magnetization curves become more rounded, and for \( n = 1 \) they become ellipses, as expected for an ohmic conductor with constant resistivity. The computed magnetization curves and susceptibilities exactly satisfy the scaling law of Sec. VI B. Namely, the same curves result, e.g., if we increase the frequency by a factor of 1000 and increase our field and current units by a factor of \( 1000^{1/(n-1)} \).

For \( n \gg 1 \) the upper and lower slightly curved parts of the hysteresis loops can be calculated by assuming that the current has saturated to \( |J| = J_c \), see Sec. V. This yields \( \nabla \times E = -\dot{\mathbf{B}}, \) thus \( |E(r,t)| = (r/2)|\dot{B}_z| = (r/2)|2\mu_0 H_0'H_0''|^{1/2} \). With \( |J_r(t)| = J_c |E(r,t)/E_r|^{1/\nu} \) one then obtains the magnetic moment of the upper and lower branches of \( m(H_a) \),

\[
m(H_a) \propto m_{\text{sat}}(1 - H_0^2/H_0^2)^{1/(2\nu)}
\]

with the saturation value \( m_{\text{sat}} \) from Eq. (25) with \( B_a = \mu_0 H_0 \) inserted. This approximation indeed fits the computed curves within line thickness in the interval \( 0 \leq H_a < H_0 \). Its derivation shows that the slight decrease of the magnetic moment \( m(H_a) \) following the steep increase of the virgin curve, is not an indication of flux creep but is due to the decrease of the ramp rate \( \dot{B}_z = \mu_0 \dot{H}_a \) after \( H_a(t) \) has reached its maximum.\(^38\)

D. Nonlinear susceptibilities

Figures 12 and 13 show the nonlinear susceptibilities \( \chi(H_0) = \chi' - i\chi'' \) obtained from the magnetization loops of Figs. 10 and 11 for the disk and a ring for creep exponents \( n = 3, 5, 11, \) and 51 in the current-voltage law \( E(J,B) = E_r[J/|J_c(B)|]^{n} \). With decreasing \( n \) (increasing creep) \( \chi'(H_0) \) is reduced and \( \chi''(H_0) \) enhanced at all ac amplitudes \( H_0 \), and the maximum of \( \chi''(H_0) \) increases and is shifted to lower \( H_0 \). Accounting for the decrease of \( J_c(B) \) with increasing local field \( B \), also modifies the susceptibility, cf. the dashed curves in Figs. 12 and 13: At large amplitudes \( H_0 \gg H_p \) both \( \chi' \) and \( \chi'' \) are reduced, while near \( H_p, \chi' \) is reduced but \( \chi'' \) enhanced, and the maximum of \( \chi'' \) (the dissipation maximum) is shifted to lower ac amplitudes.

VIII. SUMMARY AND CONCLUSIONS

The magnetization curve and nonlinear ac susceptibility of narrow type-II superconductor rings in a perpendicular field are derived for constant critical current density \( J_c \) (Bean model), Eq. (20). This result is then generalized (a) to rings of arbitrary hole diameter, (b) to finite resistivity causing flux creep, (c) to nonconstant \( J_c(B) \) (e.g., the Kim model), and (d) to thick rings and disks. The exact formulation of this
FIG. 12. Nonlinear susceptibility $\chi(H_0) = \chi' - i \chi''$ of a disk. Same case as in Fig. 10, but for creep exponents $n = 3, 5, 11,$ and 51. $H_0$ is in units $J_d$. Solid lines: $J_c =$ const. Dashed lines: $J_c(B) = J_c(0)/(0.8 + |B|)$ with $B$ in units $\mu_0 J_d$. Top: $\chi'$ and $\chi''$ versus the ac amplitude $H_0$ of $H_a(t) = H_0 \sin(\omega t)$, $\omega = 2 E_c/(\mu_0 J_d a)$. Bottom: Polar plot $\chi'$ versus $-\chi''$. The dashed lines show the Bean limit for thin disks and narrow rings.

A general problem is given in terms of an integral equation for the current density $J(r, z, t)$ in thick rings (or disks, tubes, cylinders) in axial applied field $H_a(t)$, Eq. (28), and for the sheet current $J_s(r, t)$ in thin rings, Eq. (30). As examples, the current and field profiles, magnetization curves, and nonlinear susceptibilities of thin disks and rings in cycled $H_a(t)$ are computed, accounting also for flux creep and for field dependent $J_c(B)$. Explicit expressions are given for the initial slope (42), penetration field (48), and saturation value (25) of the magnetization curve as functions of the hole radius $a_1$ and creep exponent $n$.

Inspection of the equation for $J(r, z, t)$ reveals a scaling law which states that for any current-voltage power-law $E \propto J^n$ the solutions for currents, fields, and magnetic moments remain the same when the time unit and the amplitude of the applied magnetic field are changed simultaneously. In particular, the nonlinear susceptibility in a periodic field $H_a(t) = H_0 \sin(\omega t)$ may be written as $\chi(\omega, H_0) = f(H_0)/\omega^{1n + \omega/H_0^2}$, where $\sigma = n - 1$ and $f$ is a universal function depending only on the geometry. This means that $\chi = \chi(\omega)$ for $n \to 1$ (Ohm) and $\chi = \chi(H_0)$ for $n \to \infty$ (Bean).

The distribution of $B(r, t)$ and $J(r, t)$ inside a real, three-dimensional type-II superconductor calculated from Eq. (28), may be used to compute the observed “giant magnetostriction” or “suprastriction.” This magnetomechanical effect is caused by surface screening currents and the Lorentz force density $B \times J$ exerted by the local current density $J$ on the flux lines and then transferred to the atomic lattice via flux-line pinning. The method may also be used to calculate the levitation force on cylindrical (or any rotationally symmetric) superconductors above or below a magnet, and the frequency shift and damping of vibrating superconductors in a magnetic field. The ac susceptibility of narrow rings has recently been measured in detail and closely fits formula (20).

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Such parallelograms are obtained also in rings with a weak link (Ref. 24) and in other models; see the discussion by Gilchrist (Ref. 25 and the references therein).

33 J. Gilchrist and C.J. van der Beek, Physica C 231, 147 (1994).
39 A. Gurevich (private communication).
42 J. Gilchrist (private communication).