Nonlinear current flow in superconductors with restricted geometries

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We calculate two-dimensional steady-state distributions of transport electric field \( E(x,y) \) and current density \( J(x,y) \) in superconductors with restricted geometries, such as films with macroscopic planar defects, faceted grain boundaries, current leads, flux transformers, and microbridges. We develop a hodograph method, which enables us to solve analytically Maxwell’s equations for \( E(x,y) \) and \( J(x,y) \), taking account of the highly nonlinear \( E-J \) characteristics of superconductors \( E = E_c(J/J_c)^n, n \gg 1 \). Based on this approach, a very effective numerical method of solving the nonlinear Maxwell equations was also developed. We show that nonlinear current flows in restricted geometries exhibit orientational current-flow domains separated by domain walls of varying width, which remain different from the discontinuity lines of the Bean model, even in the critical state limit \( n \to \infty \). The nonlinearity of \( E(J) \) gives rise to new length scales for \( E(x,y) \) and \( J(x,y) \) distributions, strong local enhancement of \( E(x,y) \) and long-range electric-field disturbances around planar defects on the scale \( L_e \sim anx \) much greater than the defect size \( a \). For instance, a planar defect of length \( a > d/n \) in a film of thickness \( d \) produces a narrow \((-d/\sqrt{n})\) magnetic-flux jet (domain of high electric field), which spans the entire current-carrying cross section. As a result, even small defects \( (a \sim d/n) \), which occupy only a small fraction of the geometrical cross section, give rise to significant peaks of voltage and dissipation. This nonlinear current blockage by planar defects (high-angle grain boundaries, microcracks, etc.) essentially affects the global \( E-J \) characteristics and critical currents in superconductors.

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I. INTRODUCTION

For type-II superconductors in a high magnetic field \( B \), steady-state dissipation arises as a consequence of vortex motion. Resistive response is regulated by collective flux dynamics and pinning processes, and is characterized by the local electric field-current density \( (E-J) \) relation. Figure 1 shows a typical \( E-J \) characteristic, which is highly nonlinear in the flux creep regime \( J < J_c \) and becomes linear in the flux-flow state \( J > J_c \). This \( E-J \) constituent relation can be viewed as mesoscopic, in the sense that it describes an average response, due to many vortices, over scales greater than the characteristic pinning length \( ( \text{the Larkin length} 1^{-3}) \). On the other hand, the local \( E(J,T,B) \) relation does not reflect macroscopic obstructions that affect the global \( E(J) \) characteristics on the scales \( \gg L_e \). For instance, such common current-blocking obstacles as grain boundaries, microcracks, and macroscopic second-phase precipitates in high-temperature superconductors (HTS) give rise to highly inhomogeneous (often percolative) current distributions that have been revealed by magneto-optical imaging.\(^5\)\(^-\)\(^7\) These current inhomogeneities cause strong local enhancement of the electric field near macroscopic defects, which thus become a significant current-limiting factor in HTS polycrystals,\(^8\)\(^9\) and YBa\(_2\)Cu\(_3\)O\(_7\) (YBCO) coated conductors.\(^10\) This fact poses a fundamental problem of calculating the macroscopic distributions of electric field and currents, and the global \( E-J \) characteristics of inhomogeneous HTS that we regard as highly nonlinear conductors with local \( E-J \) characteristics shown in Fig. 1. In general, the relation between the global \( \bar{E}(J) \) and the mesoscopic \( E(J) \) characteristics can be rather complex, because of the percolative nature of current flow in HTS and the nonlinearity of the \( E-J \) relation.\(^11\)-\(^13\) In this paper, we unravel the macroscopic response from the mesoscopic response by calculating exact steady-state current distributions for several key percolation structures and common circuit elements, as shown in Fig. 2. We consider here various two-dimensional (2D) transport current flows in a sample connected to a dc power supply.

A general method for calculating the steady-state macroscopic transport current flow is to combine the \( E-J \) relation with the static Maxwell equations

\[
\nabla \times E = 0, \quad \nabla \times H = J(E) .
\]

(1)

A common approximation for \( E(J) \) in superconductors at \( J < J_c \) is the power-law form

\[
E = E_0(J/J_0)^n ,
\]

(2)

FIG. 1. \( E-J \) characteristic of a type-II superconductor. \( J_c \) marks the crossover from flux creep to flux flow.
The power-law exponent other limiting case

regions with $J$ In the Bean model current paths break into critical-state re-
current flow around planar obstacles in Fig. 2, for which the conditions

however the distribution of these regions depends on initial

infinity. Both the infinitesimal thickness of the

reminiscent of the critical-state model, for which

rather unsuitable for calculations of

steady-state transport current distributions for

given boundary conditions. Additionally, the account of $E$ eliminates such artifacts of the Bean solutions as the infinite extent of current perturbations around local inhomogeneities

and the zero thickness of the $d$ lines.

To account for the electric field induced by current flow, Eq. (1) should be supplemented by the constituent relation

$$E = \rho(J)J.$$ (3)

The nonlinear superconducting response is captured in the function $\rho(J)$, for example, $\rho = [J/J_0]^{n-1}E_0/J_0$ for the power law $E(J)$. Except where noted, we will assume Eq. (2) for the particular calculations performed here, although the method presented in this paper works for any $E$-$J$ relation determined by general models of thermally activated vortex dynamics $E \propto \exp[-U(J,T,B)/T]$. We limit our scope to the case of isotropic, (2D) $E$-$J$ relations. For anisotropic layered HTS, we therefore consider current flow in the $ab$ plane with the magnetic field $\mathbf{H}[\hat{c}]$ applied along the $c$ axis.

To illustrate the complicated nature of the nonlinear Maxwell’s equations, we introduce the electrostatic potential $E = -\nabla \varphi$ and recast Eq. (1) as a scalar partial differential equation

$$\nabla \cdot [\rho^{-1}(\nabla \varphi)(\nabla \varphi)]=0.$$ (4)

For ohmic conductors ($\rho=\text{const}$), Eq. (4) reduces to Laplace’s equation, for which many solution techniques exist. For the power-law $E(J)$ relation, Eq. (4) becomes $\nabla \cdot [\nabla \varphi^{1/(1-n)/n}(\nabla \varphi)]=0$. The nonlinearity of this equation is a serious problem when calculating (either analytically or numerically) distributions of electric field in superconductors, in particular, for the common restricted geometries shown in Fig. 2. The elements in Fig. 2 can also be regarded as “building blocks” for a more general current-blocking percolative network of high-angle grain boundaries in HTS polycrystals. For ohmic conductors, some cases shown in Fig. 2 can be solved analytically using powerful methods of conformal mapping for the complex electric potential. The nonlinearity of $E(J)$ in superconductors does not allow one to use the theory of analytic functions, which greatly complicates the situation. Yet taking into account a realistic $E(J)$ relation represents a principal advantage over the critical-state model, because in this case Eq. (4) has a unique steady-state solution for given boundary conditions, which enables one to address dissipative processes induced by steady-state transport current flow in superconductors.

Recently we have developed an analytical technique of calculation of 2D distributions of electric fields and currents in superconductors with highly nonlinear $E$-$J$ characteristics. The method is based on a hodograph

FIG. 2. Characteristic cases of nonlinear current flows solvable by the hodograph method. Darker regions mark magnetic-flux jets that represent domains of enhanced electric field. The solid and dotted lines on the percolation plot show high- and low-angle grain boundaries, respectively.

where $J_0$ is a current density at a particular electric-field criterion $E_0$, Equation (2) well describes the observed $E(J)$ dependence over a very wide (several decades) range of electric fields $E$, both in low-$T_c$ (LTS) and high-$T_c$ superconductors, especially in the $ab$ plane of layered Bi-based HTS. The power-law exponent $n(T,B)$ depends on the temperature $T$ and magnetic field $B$, and typically falls into the range $3<n<40$ for HTS and $3<n<70$ for LTS. One limiting case $n=1$ corresponds to the ohmic response of a superconductor above the irreversibility field $B_i(T)$. The other limiting case $n \to \infty$ describes a stepwise function $E(J)$, reminiscent of the critical-state model, for which $E=0$ for $J<J_c$ and $J=J_c/E$ for $J>J_c$. This model has been widely used for calculations of magnetization and transport current distributions and magnetic moments of superconductors. In the critical-state model current flow exhibits characteristic discontinuity lines ($d$ lines), along which the vector $J(x,y)$ abruptly changes direction. In addition, finite-size defects can perturb $J(x,y)$ over an infinite range, as it occurs in the case of magnetization current flow past a cylindrical void, for which parabolic $d$ lines extend to infinity. Both the infinitesimal thickness of the $d$ lines and the infinite spatial extent of current perturbations result from the assumption of zero resistivity for the idealized Bean’s $E$-$J$ characteristic over the interval $0<J<J_c$.

The above features of the critical-state model make it rather unsuitable for calculations of steady-state transport current flow around planar obstacles in Fig. 2, for which the current-carrying cross section varies along a superconductor. In the Bean model current paths break into critical-state regions with $J=J_c$ and subcritical regions with $0<J<J_c$. However the distribution of these regions depends on initial conditions (see, e.g., Ref. 18) and cannot be unambiguously calculated by solving the steady-state equations $\text{div } J=0$ and $J \leq J_c$. Thus Eq. (1) for the Bean model represent an ill-defined mathematical problem, which reduces to the only condition of current continuity, $\text{div } J = 0$, provided that $J \leq J_c$. For a given sample geometry, these conditions can in principle be satisfied by many different transport current distributions. Such multiple solutions occur because the Bean model ignores the second Maxwell equation $\nabla \times E = 0$, which requires the account of the nonlinear $E$-$J$ characteristics (2). The incorporation of the $E$-$J$ material relation into Eq. (1) eliminates multiple solutions of the Bean model, fixing a unique steady-state transport current distribution for given boundary conditions. Additionally, the account of $E$ eliminates such artifacts of the Bean solutions as the infinite extent of current perturbations around local inhomogeneities and the zero thickness of the $d$ lines.

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transformation $^{25,26} \varphi(\mathbf{r}) \to \varphi(\mathbf{E})$, which converts the nonlinear Eq. (4) into a linear equation for $\varphi(\mathbf{E})$ for any $J(\mathbf{E})$. Within this hodograph representation, there are no distinctions between the linear ($n = 1$) and nonlinear ($n \neq 1$) Maxwell’s equations, therefore the same eigenfunction expansion and matrix inversion techniques previously applied to ohmic conductors can now be extended to the case of superconductors. This method proves very efficient in numerical simulations as well, allowing us to develop fast algorithms that take into account exactly both the nonlinearity of $E(J)$ and singularities in $E(x,y)$ near sharp edges. In previous publications we used the hodograph technique to obtain some exact solutions for dc current flows in superconductors with defects. $^{22-24}$ These solutions revealed new current-flow patterns and length scales caused by the strong nonlinearity of $J(\mathbf{E})$ and determined applicability limits of the critical-state model. In particular, the hodograph solutions show that current flow in superconductors generally breaks into current flow domains separated by current-domain walls of varying width, whose internal structure remains different from the well-known discontinuity lines ($d$ lines) $^{6,16,26}$ of the Bean model even in the critical-state limit $n \to \infty$. Another feature of nonlinear current flow in superconductors with macroscopic defects is a strong enhancement and long-range disturbances of the electric field on finite scales much larger than the defect size. The long-range disturbances of $E(x,y)$ become crucial for restricted geometries in Fig. 2, giving rise to essential size effects that dominate the observed global-transport properties.

This paper is devoted to the detailed analysis of nonlinear current flow for restricted geometries and is organized as follows. Section II presents a qualitative discussion of characteristic length scales resulting from the highly nonlinear $E(J)$ dependence. Section III gives a summary of the hodograph transformation technique and corresponding solutions. In Secs. IV–VII, four current-flow geometries are considered: the point source, the bridge, the current lead or elbow, and a planar defect in a film. We obtain exact solutions for the current streamlines and electric-field distributions. These are analyzed in detail, relegating technical details to the appendixes. In Sec. VIII, we consider the structures of vortex jets, current-flow domains, and domain walls, whose structure can be uncovered using the hodograph technique. Section IX is devoted to the global current-voltage characteristics and dissipation in superconductors with planar defects. In Sec. X, we conclude with a discussion.

II. LENGTH SCALES OF VORTEX JETS IN RESTRICTED GEOMETRIES

Before analyzing exact hodograph solutions, we first discuss qualitative manifestations of the strong nonlinearity of $E(J)$ in current flow around planar defects in infinite-size superconductors. Features of these solutions obtained in Refs. 22 and 23 will be crucial for determining nonlinear current flow in the restricted geometries in Fig. 2. Shown in Fig. 3 is a solution for the current streamlines around a planar defect of length $2a$ in an infinite superconductor. $^{22,23}$ The nonlinearity of $E(J)$ results in a long-range disturbance of the electric field on scales much larger than the defect size $a$, in stark contrast to a short-range decay of disturbances of $E(x,y)$ and $J(x,y)$ on the length $\sim a$ in normal metals. Another distinctive feature of superconductors is the strong anisotropy of perturbations of $E(x,y)$, which decay on very different length scales along ($L_\parallel$) and across ($L_\perp$) the direction of current flow

$$L_\perp \sim an, \quad L_\parallel \sim a\sqrt{n}. \quad (5)$$

The relation $L_\perp \sim \sqrt{n} L_\parallel \gg L_\parallel$ is characteristic of superconductors in the critical state ($n \gg 1$). $^{13,23}$ The transverse scale $L_\parallel$ can be evaluated from the following consideration. The defect blocks current flow on the length $a$, forcing the current $aJ_0$ to redistribute around the defect on the scale $\sim L_\perp$. In the region $x < L_\perp$ the mean current density and the electric field thus increase to $J_m \sim (1 + a/L_\perp) J_0$ and $E_m \sim (1 + a/L_\perp)^n E_0$, respectively. The decay length $L_\perp$ is defined by the condition $E_m \sim E_0$, thus $L_\perp \sim na$. The decay length $L_\parallel \sim a\sqrt{n}$ along current flow then follows from the general relation $L_\perp \sim \sqrt{n} L_\parallel$. $^{23}$ Therefore, strong ($n \gg 1$) nonlinearity of $E(J)$ greatly increases the spatial scales of electric-field perturbations. Note however that $E(x,y)$ can also vary on scales much smaller than $a$, as it occurs in the narrow current domain walls that replace the sharp $d$ lines of the Bean model.

As seen from Fig. 3, the disturbances of $E(x,y)$ are mostly localized in long domains of enhanced electric field of length $\sim L_\perp$ and of width $\sim L_\parallel$, sandwiched between regions $\sim L_\parallel$ of strongly depressed electric field. Because of the reflection symmetry of the current pattern in Fig. 3, only one quadrant is shown. The physical meaning of the electric-field domains in Fig. 3 becomes more transparent by consid-
erating the distribution of vortex velocities $v = -e[\mathbf{E} \times \mathbf{B}]/B^2$ around the defect in a strong magnetic field $B$ when the self-field effects are negligible and the local $B(x,y)$ equals the applied field $B_a$. Then the elongated domains of strong electric field $E(x,y) > 1.5E_0$ in Fig. 3 correspond to a channel of preferential magnetic-flux motion along the $x$ axis. The flux is sucked from one of these electric-field domains into the edge of the defect and then gets injected from its opposite edge into the second domain, which therefore can be regarded as a magnetic-flux jet in a strongly pinned vortex structure. These jets are situated between stagnation regions of nearly motionless flux (in Fig. 3 only one flux jet and half of the stagnation region are pictured). We emphasize that both the macroscopic flux jets and stagnation regions result from the geometry of current flow and the strong nonlinearity of $E(J)$, but are not due to enhanced or reduced local flux pinning. As shown below, both the flux jets and stagnation regions are characteristic of nonlinear current flow in superconductors of restricted geometries. For instance, nonlinear current transport of polycrystals in Fig. 2 can be formulated as a percolation of magnetic flux along a discontinuous network of high-angle grain boundaries connected by vortex jets. Notice that dissipation and resistance are mostly localized within comparatively narrow vortex jets that thus become peculiar “hot spots” in polycrystals.

Because of the large length of flux jets around defects for $n \gg 1$, the effects of the sample geometry in superconductors become much more pronounced than for Ohmic conductors. As an illustration, we consider a planar defect in a film of thickness $d$ or an equivalent case of a chain of planar defects, which models faceted grain boundaries shown in Fig. 2. The length of the flux jet becomes larger than $d$ for $L_\perp > d$ or

$$a > d/n. \quad (6)$$

If this rather weak condition is satisfied (given typical $n \sim 20–30$ in HTS and $30–70$ in LTS), then even a small defect ($a \ll d$) can produce a flux jet that effectively blocks the current-carrying cross section. This results in a localized voltage step at the defect and excess dissipation. The width of the flux jet $L_\perp$ can be estimated from the relation $L_\perp/L_a \sim \sqrt{n}$, where the length of the jet $L_\parallel \sim d$. Hence we obtain length scales of the jet in restricted geometries

$$L_\perp \sim d, \quad L_\parallel \sim d/\sqrt{n}. \quad (7)$$

The strong nonlinearity of $E(J)$ results in a peculiar “pinch effect,” manifesting itself as a contraction of the flux jet as $n$ increases. In the critical-state limit $n \to \infty$ the jet turns into a line.

The contraction of the vortex jet with increasing $n$ is accompanied by a strong increase of vortex velocities and thus electric field $E(x,y)$ in the jet as compared to the electric $E_0$ for the uniform current flow far away from the defect $|x| \gg d/\sqrt{n}$. This follows from the current-continuity condition $J_0 d = J_m (d-a)$, where $J_m$ and $E_m = E_0 (J_m/J_0)^n$ are the current density and the electric field in the jet on the side of the film opposite to the defect. Hence,

$$E_m \sim \left(\frac{d}{d-a}\right)^n E_0. \quad (8)$$

For $n = 30$, we obtain $E_m \approx 17.5E_0$ for a 10% defect ($a = 0.1d$) and $E_m \approx 237E_0$ for a 20% defect ($a = 0.2d$). The enhancement of the electric field and dissipation in the flux jet manifests itself in a significant excess voltage $\Delta V \sim E_m L_\parallel$.

$$\Delta V \sim \frac{dE_0}{\sqrt{n}} \left(\frac{d}{d-a}\right)^n. \quad (9)$$

Another distinctive feature of the current flow observed in Fig. 3 are the regions where the nearly constant current density $J = J_c$ sharply changes direction. These orientational current-flow domains are separated by comparatively narrow current-domain walls, reminiscent of the $d$ lines between regions of different critical-flow orientation in the Bean model. However, there are important differences between the current domain walls described by exact solutions of the Maxwell equations and the phenomenological $d$ lines. First of all, the current domain walls have an internal structure and a varying width $\sim d/\sqrt{n}$, which depends both on the $n$ value and the geometry of current flow. For instance, for current flow past a planar defect in Fig. 3, the width of the current domain walls increases with the distance from the defect. It is the broadening of the domain walls that provides the decay of current perturbations on a finite length $L_\perp \sim \sqrt{n}$. Moreover, the current domain walls described by exact hodograph solutions remain different from the $d$ lines even in the critical-state limit $n \to \infty$, for which the current flow near domain walls exhibits no discontinuities in the tangential components of $J(x,y)$ and $E(x,y)$. The spatial scales of the orientational current-flow domains for the infinite geometry in Fig. 3 are of the order $L_\perp \sim \sqrt{n}$. As shown below, the sizes of these orientational domains for restricted geometries become of the order of the sample dimension $d$ both along and across current flow.

We conclude this section with a brief discussion of the transient time scales $t_0$, on which the steady-state transport current distributions set in after the dc power supply was turned on. The time constant $t_0$ is determined by magnetic-flux diffusion and is inversely proportional to a final dc electric field $E_0$, in a superconductor

$$t_0 \sim \mu_0 s J_c L_\perp^2/E_0, \quad (10)$$

where $L$ is a characteristic sample size (thickness or width), $J_c$ is the critical current density, and $s = \partial \ln J_c/\partial \ln t$ is the dimensionless flux creep rate. For the power-law $E-J$ relation, we have $s = 1/(n-1)$. We illustrate Eq. (10) for a planar defect in a film, for which the transient time evolution of the current flow is sketched in Fig. 4. Here Fig. 4(a) shows an initial distribution $J(x,y)$ in which the white region carries no current, while the shaded upper part of the sample cross section is in a critical state and carries the current density $J_m = I/(d-a)$, where $I$ is the total fixed current. This unrelaxed current configuration has an infinite stagnation region (white) and is consistent with the Bean model. However
Unless specially noted, we consider in this paper a true steady-state current flow ($t \gg t_0$), thus neglecting the transient relaxation. Detailed numerical analysis of nonlinear flux diffusion in superconductors in ac magnetic fields have been performed by Brandt.28

III. HOLOGRAPH TRANSFORMATION

A. Formalism

In this work we consider a superconductor in a strong magnetic field $B_0 \gg B_p$, so that self-field and demagnetization effects are negligible. Then the constitutive relation $E = \rho(B,J)J$, which generally depends on the local $B(x,y)$, especially for $B < B_p$, can be replaced by the relation $E = \rho(B,J,J)$, which is independent of $B$. In this case the 2D distribution $J(x,y)$ can be calculated by introducing the electric potential $\psi$ and the stream function $\psi$:

$$ E = - \nabla \psi, \quad J = \nabla \psi \times \hat{z}, \quad \nabla \cdot J = 0. \quad (11) $$

The contours of the function $\psi(x,y) = \text{const}$ point in the direction of local current flow, and their density is proportional to the local current density. In Ohmic conductors, both $\varphi$ and $\psi$ satisfy the Laplace equations $\nabla^2 \varphi = 0, \nabla^2 \psi = 0$. This fact enables one to introduce the complex potential $w(z) = \varphi - i \rho \psi$ as an analytic function of the complex coordinate $z = x + iy$, obtaining various exact solutions for the 2D current flow.19,20

For superconductors ($n > 1$), the potential $w(z)$ satisfies a highly nonlinear partial differential equation and is no longer an analytic function. However, we can apply the hodograph transformation,25,26 which linearizes the equation for $w(x,y)$ by changing variables from the Cartesian coordinates $x$ and $y$ to $J$ and $\theta$ (or, equivalently, $E$ and $\theta$). The amplitude of the current density $J(x,y)$ and the corresponding flow angle $\theta(x,y)$ are defined through the relation $J e^{i\theta} = J_x + iJ_y$.

To show how the hodograph transformation works, we write Eqs. (10) in the differential form

$$ d\varphi = -E_x dx - E_y dy, \quad \rho d\psi = E_x dx - E_y dy, \quad (12) $$

where $\rho(E) = E/J$. Equation (12) can be combined in a single complex equation $d\varphi - i \rho d\psi = -E e^{-i\theta} dz$, thus

$$ \partial E_z = -E e^{-i\theta} \partial_x \varphi - i \rho \partial_y \varphi/E, \quad (13) $$

$$ \partial E_\theta = -E e^{i\theta} \partial_x \psi - i \rho \partial_y \psi/E, \quad (14) $$

where $\sigma(E) = \partial J/\partial E$ is the differential conductivity and $\partial = \partial / \partial \alpha$. The condition $\partial^2_{E\theta} = \partial^2_{E\psi}$ gives the following relations between partial derivatives of $\psi$ and $\varphi$:

$$ \partial_\theta \varphi = -(E^2/J) \partial_\theta \psi, \quad \partial_\psi \varphi = (E \sigma J^2) \partial_\psi \psi. \quad (15) $$

Equating the mixed derivatives $\partial^2_{E\theta} = \partial^2_{E\psi}$ for $\psi$ and $\varphi$, we obtain the following linear equations for $\psi$ and $\varphi$ valid for any nonlinear dependence $J(E)$:

$$ \frac{\sigma E}{J^2} \partial^2 \psi \quad \partial \left( \frac{E^2 \partial \psi}{J \partial E} \right) = 0, \quad (16) $$

FIG. 4. Evolution of transport current distribution for a film with a crack toward the steady state after switching on the dc power supply. Different shades of gray quantify magnitudes of electric field (in three-level scale), and arrows show the direction and magnitude of local current density $J$. (a) shows initial unrelaxed state of nearly constant $J$ above the crack. (b) shows a transient state, for which magnetic-flux diffusion tends to make the electric field more uniform across the sample, and the stagnation region and the flux jet are formed. (c) shows the final dc state, which is a cartoon of an exact hodograph solution obtained in Sec. VII. Here the darkest and the white areas mark the magnetic-flux jet and the stagnation region, respectively.
\[
\frac{J}{E^2} \partial^2 \varphi + \partial \left( \frac{J^2}{\sigma E} \partial \varphi \right) = 0. \tag{17}
\]

Equations (16) and (17) simplify considerably for the power law \( E(J) \), giving

\[
\frac{J^2}{n} \frac{\partial^2 \psi}{\partial J^2} + J \frac{\partial \psi}{\partial J} + \frac{\partial^2 \psi}{\partial \theta^2} = 0, \tag{18}
\]

\[
nE^2 \frac{\partial^2 \varphi}{\partial E^2} + E \frac{\partial \varphi}{\partial E} + \frac{\partial^2 \varphi}{\partial \theta^2} = 0. \tag{19}
\]

In Eq. (18) we can change variables \( E = E_0(J/J_0)^n \) to express \( \psi(E, \theta) \) as a function of the electric field \( E \). This representation will be convenient later on when calculating 2D distributions of \( E(x, y) \).

Equations (18) and (19) can be solved by the separation of variables of the form \( 22,23 \)

\[
\frac{\psi(J, \theta)}{I} = A_0 + (B_0 + D_0 \theta) \left( \frac{J}{J_0} \right)^{1-n} + C_0 \theta
\]

\[
+ \sum_{m=1}^{\infty} C_m \left( \frac{J}{J_0} \right)^{\tau^+} \sin(m \theta + \phi_m)
\]

\[
+ \sum_{m=1}^{\infty} D_m \left( \frac{J}{J_0} \right)^{\tau^-} \sin(m \theta + \omega_m). \tag{20}
\]

\[
\tau^+ = \frac{1}{2} \left[ 1 - n \pm \sqrt{(n-1)^2 + 4nm^2} \right]. \tag{21}
\]

The total current \( I \) and the average sheet current density \( J_0 = I/d \), where \( d \) is the sample width are natural scales for \( \psi \) and \( J \), adopted throughout this work.

As described in Refs. 22 and 23, Eqs. (13)–(15) can be used to perform the inverse transformation from the hodograph to physical coordinates, which yields

\[
\frac{z(J, \theta)}{d} = F_0 + C_0 \left( \frac{J}{J_0} \right)^{-n} \left\{ iB_0(n-1) + D_0[1 + i(n-1) \theta] \right\}
\]

\[
- e^{i\theta} \sum_{m=1}^{\infty} C_m \left( \frac{J}{J_0} \right)^{\tau^+} \sin(m \theta + \phi_m)
\]

\[
- i \tau_m^+ \sin(m \theta + \phi_m)
\]

\[
- e^{i\theta} \sum_{m=1}^{\infty} D_m \left( \frac{J}{J_0} \right)^{\tau^-} \sin(m \theta + \omega_m)
\]

\[
- i \tau_m^- \sin(m \theta + \omega_m). \tag{22}
\]

Equations (20) and (22) constitute a complete family of exact solutions for isotropic 2D current flows. The solutions for \( \psi(J, \theta) \) and \( z(J, \theta) \) are expressed parametrically, in terms of the hodograph variables \( J \) and \( \theta \) and expansion coefficients \( A_0, B_0, C_m, D_m, F_0, \phi_m, \) and \( \omega_m \). The problem is therefore reduced to determining these expansion coefficients for particular boundary conditions.

### B. Boundary conditions

The utility of the hodograph transformation rests on two criteria. The first one is that the Maxwell equations become linear in the hodograph representation. The second criterion is equally important: the transformed boundary conditions (BC) in the hodograph plane must be simple enough to enable analytical solutions. In general, the hodograph transformation results in highly nonlinear BC’s for current flow around curved boundaries.\(^{22,23} \) However there is a wide class of physical problems like those shown in Fig. 2 for which the BC’s remain linear, which enables us to obtain exact solutions for nonlinear current flows.

The eigenfunction expansions given in Eqs. (20) and (22) are particularly convenient for the following BC in the physical plane. First, the sample edges should be straight to ensure that the current, flowing tangentially along the sample boundary, is represented by a constant flow angle \( \theta_0 \). The transformed BC then corresponds to a straight line \( \theta = \theta_0 \) in the hodograph plane. Second, the angles inscribed inside sample corners should be rational fractions of \( 2\pi \); this produces zeros of the eigenfunctions at appropriate values of \( \theta \). This latter restriction is not severe, since other eigenfunction expansions are readily obtained for different angular configurations.

To illustrate these BC’s, we consider a planar defect in a film, for which a detailed solution is given in Sec. VII. The physical representation of this geometry is shown in Fig. 5(a). Current flowing through a channel is blocked by an unpenetrable wall BC forcing the current to pass through an aperture CD. Far away from the constriction, the flow becomes uniform, with current density \( J_0 \). At point \( D \), the current density reaches a value \( J_m \). The current density possesses a singularity at point \( C \) and a stagnation point \( B \), where the current density drops to zero. Because of symmetry, we only need to solve one half of the flow problem (top half). We make point \( D \) the origin of the physical plane. \( A \) and \( E \) represent asymptotic points far from the constriction.
Along any wall, the normal component of $\mathbf{J}$ must vanish. Equation (11) indicates that the stream function will be constant along such a boundary. Since $\psi(x,y)$ contains an arbitrary additive constant, we choose $\psi|_{DE}=0$ along the line $DE$. Then along the boundary $ABC$, we set $\psi|_{ABC}=1$, as consistent with a total current $I$. Along the symmetry line $CD$, the tangential component of $\mathbf{J}$ vanishes, thus $\partial_\theta \psi|_{CD}=\partial_r \psi|_{CD}=0$.

Because of the simplicity of the BC in the physical plane, points $A$, $B$, $C$, $D$, and $E$ can be mapped immediately onto the hodograph plane, as demonstrated in Fig. 6. Current flow along the lines $AB$, $CD$, and $DE$ is characterized by $\theta=\pi/2$, while flow along $BC$ corresponds to $\theta=\pi$. At points $B$ and $C$, the current “turns a corner.” These two points are mapped onto lines in the hodograph plane: $J=0$ and $J=\infty$, respectively.

The characteristic current scales $J_0$ and $J_m$ naturally divide the hodograph plane in Fig. 6 into three regions with distinct BC. The lines separating these regions (e.g., $J=J_0$) have no special significance in the physical plane. We therefore enforce the following continuity conditions at $J=J_0$ and $J=J_m$:

$$\psi^{(1)}(J_0) = \psi^{(2)}(J_0), \quad \partial_\theta \psi^{(1)}(J_0) = \partial_\theta \psi^{(2)}(J_0),$$

where $\psi^{(1)}$ and $\psi^{(2)}$ represent the stream-function solutions in adjoining regions. These are solved by Fourier analysis in terms of the variable $\theta$. In this way, the matching conditions (23) give the equations for the hodograph expansion coefficients, which can be expressed in the standard matrix form

$$\sum_{m'=1}^{\infty} K_{m,m'} C_{m'}=G_m,$$

where $C_m$ represents the expansion coefficients in all the different hodograph regions. Thus, the values $C_m$ can be obtained by matrix inversion of Eq. (24) either analytically or numerically using any of the well-developed and fast algorithms.

IV. POINT SOURCE

An instructive geometry is the point source flowing into an infinite plane. This case models a current lead attached to the film surface, or the Corbino-disk geometry used for studies of vortex dynamics in HTS. From symmetry considerations, it is sufficient to consider a point source located at the origin flowing into a single quadrant, as shown in Fig. 7(a). The lines $x=0$ and $y=0$ then form sample boundaries. Because of radial symmetry, the point-source geometry can be solved in Cartesian coordinates for any nonlinear $E(J)$. We consider the radial coordinates $r$ and $\theta$, defined as $r e^{i\theta}=z=x+iy$. The total current $I$ flowing across the quarter-circle arc $r=\text{const}$ is given by $I=\pi r J/2$, thus

$$J = \frac{2I}{\pi r}.$$ (25)

From Eq. (11), we obtain $\nabla \psi=(2I/\pi r) \hat{\theta}$, whence $\psi=2I \theta/\pi+\text{const}$. The formulas of this section can be used for the Corbino disk by replacing $I\rightarrow I/4$.

It is illuminating to reproduce these results using the hodograph transformation. The values of $\psi$ on the vertical and horizontal sample boundaries are constants given by $\psi=0$ and $I$, respectively, as shown in Fig. 7(b). In this case all coefficients $C_m$ and $D_m$ except $C_0$ and $A_0$ in the general solution (20) vanish. We therefore associate a nonzero value of $C_0$ with point-source behavior, which often appears in more complicated geometries, as shown below. The boundary requirements are met by the solution $A_0=-1$, $C_0=2I/\pi$, or

$$\psi=-I+2I \theta/\pi.$$ (26)

From Eq. (22) with $J_0 d=I$, we arrive at the solution

$$z=e^{i\theta} \frac{2I}{\pi J},$$ (27)

which is equivalent to Eq. (25).
Equations (26) and (27) result from the current conservation and radial symmetry, thus they are independent of the form of $E(J)$. We can exploit this situation to derive results that are not easily obtained in other geometries. For example, since $J$ becomes singular at the point source, a circular flux-flow domain is nucleated around the origin. If the flux flow occurs when $J(r)$ exceeds a critical current density $J_c$, the domain radius is given by $r_c = 2I/\pi J_c$. Assuming a relation $E = \rho_F (J - J_c)$ within the flux-flow domain, the resulting dissipation is given by

$$Q = \int E \cdot J \, d^2 r = \frac{2I^2 \rho_F}{\pi} \left[ \ln \left( \frac{2I}{\pi \rho_F J_c} \right) + \pi b J_c \right].$$  

(28)

A flux-flow region thus surrounds a small current lead of radius $b < r_c$. As shown in the following sections, current flowing around a sharp corner also induces flux-flow domains.

V. BRIDGE

The bridge geometry, pictured in Fig. 8(a), is quite common in transport experiments. By symmetry, it is only necessary to solve for half of the current distribution; we focus on the left half. The dashed reflection line at the center is taken as a sample boundary. Along this line the stream function is a constant, and we set $\psi = 0$. Along the other boundary, we have $\psi = I$. In contrast to the point source, the bridge geometry has a characteristic length scale $b$ (the bridge width), and a current-density scale $J_0 = I/b$, corresponding to uniform current flow in the channel, far from the opening ($y < -b$). The point source corresponds to the limit $b \to 0$.

As shown in Fig. 8(b), the hodograph plane consists of two regions separated by the line $J = J_0$. The solutions for current flow through the bridge are given in Appendix B, including exact expressions for the hodograph coefficients in Eqs. (20) and (22). Current streamlines are shown in Fig. 9 for an ohmic conductor ($n = 1$) (a) and a superconductor with $n = 30$ (b). Similar asymptotic behaviors are observed in the two plots. In the upper-half plane, the asymptotic flow, far from the channel opening, is point-source-like, with the apparent point source located at the origin. Mathematically, the source of this behavior is in the term $2 \theta/\pi$ in Eq. (B1) and the corresponding term $(2e^{i\theta}/\pi)J/J_0$ in Eq. (B3). When $J_0 \to 0$, these terms dominate, and we recover the point-source solutions of Sec. IV. In the lower-half plane, far from the opening, the current approaches its asymptotic uniform behavior: $J = J_0 \delta^2$. As follows from Fig. 9, for the normal metal, the transition between the different asymptotic regimes is rather gradual, while for a superconductor ($n \gg 1$), the transition is much sharper and occurs mainly near the channel opening.

We now use the general formulas of Appendix B to calculate distributions of $J(x,y)$ and $E(x,y)$ along the sample surface and the central line $x = 0$. Along the vertical sample edge, adjacent to the corner $x = -b$, we have $\theta = \pi/2$ and $J > J_0$. Then Eq. (B4) gives $J(y)$ in the form

$$y = -2b \sum_{m=1}^{\infty} \frac{\tau_2^m}{\tau_2^m - \tau_2^{m-1}} \left( \frac{J}{J_0} \right)^{\tau_2^{m-1} - 1},$$  

(29)

where $\tau_2^m$ are given by Eq. (21).

The compression of current streamlines near the corner results in singularities of $E(x,y)$ and $J(x,y)$, which may produce local flux-flow behavior. Very near the corner, where $J \to \infty$, the lowest-order term with $m = 1$ in Eq. (29) dominates the sum, giving

$$J \propto |z + b|^{1/3(\tau_2^{1} - 1)}, \ E \propto |z + b|^{n/(\tau_2^{1} - 1)}.$$  

(30)

These results are valid when approaching the singular point from any direction. When $n = 1$ we obtain $J \propto E_0^2 |x + iy + b|^{-1/3}$, while for $n \gg 1$, we have $J \propto |x + iy + b|^{-1/n}$ and $E \propto 1/|x + iy + b|$. As $n$ grows, the singularity in the current density becomes weaker, vanishing as $n \to \infty$. In this limit,
the current density becomes uniform below the channel opening \((y = 0)\), such that \(J = J_0 \delta\). However, the singularity in the electric field is enhanced for large \(n\) as compared to \(n = 1\). Similar large-\(n\) singular behavior was observed previously in Refs. 22 and 23, for current flow past a small planar defect. This behavior is a general feature of flow past a sharp corner for any angle.

For finite \(n\), the current density approaches its asymptotic uniform value \(J_0\), slightly below the channel opening \((y \leq -b)\). In this “tail” region, we have \(J_0 - J(y) \ll J_0\), so the convergence of the sum in Eq. (29) is determined by the region \(m \gg 1\). Taking large-\(m\) asymptotics of \(\tau_{2m}\) from Eq. (21), we rewrite Eq. (29) as

\[
y = W_n - \sum_{m=1}^{\infty} \frac{b(J_0/J)^{2m+1}}{\pi \sqrt{nm}}
= W_n + l_b \ln[1 - (J_0/J)^{2+\frac{1}{n}}],
\]

thus

\[
J = J_0 + \frac{J_0}{2 \sqrt{n}} e^{-(y-W_n)/l_b}. \tag{32}
\]

The decay length \(l_b\) and the shift \(W_n\) are given by

\[
l_b = b / \sqrt{\pi n}, \tag{33}
\]

\[
W_n = -b \sum_{m=1}^{\infty} \left[ \frac{4 \tau_{2m}^+}{\pi (\tau_{2m}^+ - \tau_{2m}^-)(\tau_{2m}^+ - 1)} - \frac{1}{\pi \sqrt{nm}} \right]. \tag{34}
\]

The value \(W_n\) is weakly \(n\) dependent. The length \(l_b\) is similar to the characteristic scale \(L_1\) in Eq. (7). When \(n \to \infty\), the tail disappears \((l_b \to 0)\), so that \(J = J_0\) for all \(y < 0\).

Now we calculate \(J(x)\) along the horizontal sample edge, adjacent to the corner, where \(x < -b, y = 0\). Away from the small (for \(n \gg 1\)) singular region near the corner, the current density falls into the range \(J < J_0\), with a current-flow angle \(\theta = \pi\). In this case the distribution \(J(x)\) is obtained from Eq. (B3) in the form

\[
x = -\frac{2I}{\pi J} + \sum_{m=1}^{\infty} \frac{4 b(-1)^m \tau_{2m}^+(J/J_0)^{\frac{1}{n}}}{\pi (\tau_{2m}^+ - \tau_{2m}^-)(\tau_{2m}^+ - 1)}.
\]

Far away from the bridge, where \(J \ll J_0\), the first (point-source) term dominates, giving \(x = -2I / \pi J\). Corrections to this description correspond to leading-order terms in the summation.

Now we calculate the electric-field distribution along the central line \(x = 0\), where \(\theta = \pi/2\) and \(J < J_0\). From Eq. (B3) we obtain the following relation for the distribution \(y(E)\):

\[
y = \frac{2 b}{\pi} \left[ \frac{E}{E_0} \right]^{1/n} - \sum_{m=1}^{\infty} \frac{4 b \tau_{2m}^+(E/E_0)(\tau_{2m}^{+1/n})}{\pi (\tau_{2m}^+ - \tau_{2m}^-)(\tau_{2m}^+ - 1)}.
\]

Results for \(E(y)\) obtained from Eq. (36) are shown in Fig. 10(a). For \(y \leq 0\), the electric field quickly attains its asymptotic uniform flow state \(E = E_0\). Equation (32) describes the behavior of the tail in this region with a modified \(W_n\). For \(y \geq 0\), the electric field converges to its asymptotic point-source behavior \(E = E_0 (2b/\pi y)^n\). Near the opening, where \(y = 0\), \(E\) crosses over between these two asymptotic regimes. In this vicinity, \(E\) is also affected by the nearby corner singularity.

We use Eq. (36) to calculate the excess voltage \(\Delta V = V - V_0 L\), defined as the difference between the total voltage signal and the assumed voltage drop along the bridge. This addresses a common situation in transport measurements, where the edge singularities and nonuniform distribution of \(E(x,y)\) near the bridge opening are not taken into account by the conventional interpretation, which assumes a uniform distribution \(E(x,y) = E_0\) inside the bridge \((y < 0)\) and \(E(x,y) = 0\) for \(y > 0\). To calculate the \(\Delta V\), we consider two current leads attached to either side of the sample, along the center (mirror) line \(x = 0\) [see Fig. 8(a)]. For simplicity, the leads are placed at an infinite distance from the bridge element \((y \to \pm \infty)\). We assume that the bridge is long enough that the two openings do not interact.

The excess voltage \(\Delta V\) is obtained from the integral

\[
\Delta V = 2 \int_{-\infty}^{0} (E - E_0) dy + 2 \int_{0}^{\infty} E dy
d = 2 \int_{E_0}^{E_1} (E - E_0) \frac{\partial y}{\partial E} dE + 2 \int_{E_1}^{0} E \frac{\partial y}{\partial E} dE, \tag{37}
\]

where the function \(y(J)\) is given by Eq. (36), \(E_1\) is the value of \(E\) for which \(y(E_1) = 0\), and the factor 2 accounts for two
ends of the bridge. We may also express $\Delta V$ in terms of an effective excess length $\Delta L = \Delta V / E_0$, which represents the length of a bridge element needed to produce the same excess voltage signal $\Delta V$, assuming a uniform electric field $E_0$. In Fig. 10(a), the excess length $\Delta L$ is shown for $n = 30$. The excess voltage $\Delta V$ as a function of $n$ is shown in Fig. 10(b). The rapid drop off of the electric field for point-source flow ($E \approx y^{-n}$) causes $\Delta L$ and $\Delta V$ to decrease with increasing $n$. For $n \to 1$, the slower drop off $E \approx 1/y$ produces a logarithmic singularity in the voltage integral, Eq. (37). For the bridge geometry, the ohmic case is thus particularly sensitive to the placement of voltage taps.

VI. CURRENT LEAD

The current-lead or elbow geometry is shown in Fig. 11(a), which also describes the flux transformer configuration. The current lead contains two current-density scales $J_1 = l/d$ and $J_2 = l/b$, which correspond to current flow in the large and small channels, far away from the corner. The anisotropy parameter is defined as $r = b/d$, with $0 < r < 1$. The hodograph plane of the elbow geometry consists of three regions, separated by the lines $J = J_1$ and $J = J_2$, as shown in Fig. 11(b). This problem is more complicated than that for the bridge, since it involves an additional hodograph region. Nevertheless, the current-lead geometry can be solved exactly, as shown in Appendix C.

Several limiting cases can be identified. The case $r = b/d = J_1/J_2 = 1$ corresponds to a symmetric elbow, for which the hodograph region (2) vanishes. In a different limit, we take the upper film edge to infinity ($d \to \infty$, $r \to 0$) to obtain the bridge geometry. Since the current density $J_1$ then goes to zero, the lower hodograph region (1) (with $0 < J = J_1$) disappears. Region (2) is now bordered by the line $J = 0$, and we set the coefficients $D_{2m}^{(2)}$ to zero in order to keep the stream function finite. In this way, we recover the bridge solution of Sec. V and Appendix B.

Current streamlines for the elbow geometry are shown in Fig. 12. For $n \approx 1$, the streamlines are compressed near the corner, and a singularity appears in some region very near the corner. As $n$ increases, the singular region shrinks in size, vanishing in the limit $n \to \infty$. More remarkably, current flow breaks up into characteristic flow domains, including uniform flow domains, where $J = \text{const}$, and a point-source flow domain, for which $J \approx 1/|z|$. This point-source behavior originates primarily in hodograph region (2), whose solution contains the leading terms proportional to $C_0$ in Eqs. (20) and (22). By contrast, such orientational flow domains do not occur in normal conductors, where streamlines turn smoothly around the corner.

We use the exact solution of Appendix C to calculate electric field and current-density distributions $E(x)$ and $J(x)$ along the bottom and top edges of the main channel. These distributions are of prime interest in flux transformer experiments in which voltage taps are placed along “primary” (bottom) and “secondary” (top) edges, respectively. Along the primary edge (see inset of Fig. 13) we have $J > J_1$. Very near the singular point ($x + iy = -b$) we enter the high-current regime $J > J_2 > J_1$. Since $\theta = \pi$ along the primary line, Eq. (C6) gives $J(x)$ in the form

$$x = -b + \sum_{m=1}^{\infty} \frac{4b \tau_{2m}^{(2)}}{\pi(\tau_{2m}^{(2)} - \tau_{2m}^{+})(\tau_{2m}^{-} - 1)} \left( \frac{J}{J_2} \right)^{-\tau_{2m}^{-} - 1}. \quad (38)$$
The exponential convergence of the electric field over the length \( l_e = d/\sqrt{n} \) is characteristic of confined geometries, as discussed in Sec. II. We contrast this result with the radial current flow in the upper half of the bridge geometry in Fig. 9, for which asymptotic convergence to point-source flow away from the origin occurred as a power law. The second length scale \( W_n \) has particular physical significance for \( n \to \infty \), when

\[
W_\infty = \frac{2d}{\pi} - \sum_{m=1}^{\infty} \frac{4d(-1)^m(b/d)^{4m^2}}{\pi(4m^2-1)}.
\]  

(42)

For \( b/d < 0.88 \), the series may be truncated at one or two terms with excellent accuracy, giving \( W_\infty \approx 2d/\pi \) if \( b/d \). For \( b/d > 0.88 \), we can set \( b/d \to 1 \) in Eq. (42), obtaining \( W_\infty \to d \) after summation.

The meaning of \( W_\infty \) can be understood as follows. For \( n \approx 1 \), the electric field converges rapidly to its asymptotic value \( E_1 \), away from the corner. As \( n \to \infty \), the decay length \( l_e \to 0 \) vanishes, and the perturbations of the electric-field perturbations disappear in the regions \( |x| > W_\infty \), where the uniform flow state sets in: \( E = -E_1 \hat{y} \). For finite \( n \), the electric field becomes nonuniform, and \( W_n \) marks the onset of the exponential decay of the electric-field disturbance. The main disturbance in \( E(x,y) \) therefore occurs in a rectangle \( W_n \times d \), as shown in Fig. 12(b).

Next, we investigate the electric field and current distributions along the secondary sample edge. Along this line we have \( J = J_1 \) and \( \theta = \pi \). Thus Eq. (44) gives

\[
x = \frac{4d \tau_{2m}[-1 + (-1)^m p r^{2m}]}{\pi(\tau_{2m} - \tau_{2m}^+) (\tau_{2m}^+ - 1)} \left( \frac{J}{J_1} \right)^{\tau_{2m}^+ - 1}.
\]  

(43)

Near the stagnation point \( x + iy = id \), where the electric field vanishes, the term with \( m = 1 \) in Eq. (43) dominates, giving

\[
E \approx |x + iy - id|^{n/(\tau_2^+ - 1)}.
\]  

(44)

When \( n = 1 \), we have \( J \approx E \cdot E_1 \cdot x + iy - id. \) Therefore \( J \approx |x + iy - id|^{1/3} \) and \( E \approx |x + iy - id|^{n/3} \).

Distributions of \( E(x) \) along the secondary edge given by Eq. (43) are shown in Fig. 13. For \( n \approx 1 \), the electric field drops exponentially at \( x < -W_n \) converging to a step function at \( x = -W_n \) when \( n \to \infty \). In the tail region (\( x \approx -W_n \)), where \( E \approx E_1 \), Eq. (43) can be estimated using the same technique that led to Eq. (40) for the primary edge. We obtain exactly the same result, but with a new expression for \( W_n \). In the limit \( n \to \infty \), this reduces to the previous expression for \( W_\infty \), as given in Eq. (42).

Using Eqs. (39) and (43), we can compute quantities of interest for the flux-transformer geometry, including primary and secondary voltage distributions \( V_p \) and \( V_s \), as defined in the inset of Fig. 14. We consider the configuration where primary and secondary electrodes are placed at the same positions on the top and bottom sample edges. As apparent in Fig. 13, a large enhancement of the electric field occurs within a distance \( W_n \) of the current lead on the primary edge, while a suppression occurs along the secondary edge. We
FIG. 14. Ratio of secondary to primary voltage signals \( V_s / V_p \) vs \( n \) for the transformer geometry with \( r = b/d = 0.1 \). Curves correspond to three different placements of voltage taps \( x_0 \) at the fixed distance \( \delta x = 0.2d \) between the taps: (a) \( x_0 = 0.8d \), (b) \( x_0 = 0.55d \), (c) \( x_0 = 0.3d \).

therefore expect \( V_s / V_p \) to decrease as the voltage taps are moved closer to the current lead, due to simple geometrical effects.

The voltage drops \( V_p \) and \( V_s \) between the points \(-x_0 - \delta x\) and \(-x_0\) are computed from

\[
V = \int_{-x_0-\delta x}^{-x_0} E(x)dx = E_1 \int_{J_L} J \, n \frac{\partial x}{\partial J} dJ, \tag{45}
\]

where \( J_L = J(-x_0 - \delta x) \) and \( J_R = J(-x_0) \). The distribution \( x(J) \), appearing in the second equality, is taken from Eqs. (39) and (43).

In Fig. 14, we show the results for three different calculations of \( V_s / V_p \). Curve (a) corresponds to the case where both voltage taps are placed far from the current lead, so that \( x_0 > W_\infty \) (see inset). For curve (b), the taps are placed on either side of \( x = -W_\infty \). Curve (c) shows the case where taps are placed near the current lead, with \( x_0 + \delta x < W_\infty \). The ratio \( V_s / V_p \) is strongly suppressed when the measurements are performed close to the current lead. As \( n \) increases, the ratio \( V_s / V_p \) decreases. Such behavior does not signal a loss of vortex coherence between the primary and secondary edges due to a ‘‘decoupling transition,’’ but simply results from geometrical effects. These effects can be avoided by placing voltage taps outside the region of strong electric-field perturbations.

Flux-transformer experiments are usually performed on anisotropic HTS with the \( c \) axis parallel to \( \hat{y} \). In this case current flows in strongly anisotropic ac or bc planes, thus the region of strong perturbations of \( E(x,y) \), becomes highly elongated. This problem cannot be treated exactly within the present analysis,\(^{21}\) but we can estimate \( W_\infty \) for the anisotropic case as \( W_\infty \sim d \sqrt{\rho_y / \rho_x} \). The latter estimate is exact for ohmic conductors and might be qualitatively valid for anisotropic HTS. For Bi-based HTS the resistivity ratio can be of the order \( 10^3 \), thus the electric-field disturbance gets elongated in the \( \hat{x} \) direction by a factor \( \sim 100 \). In this case, anomalous geometrical effects, demonstrated in Fig. 14, could occur far away from the current leads.

VII. PLANAR DEFECT IN A FILM

The constriction geometry, considered here, is characteristic of transverse microcracks that frequently occur in HTS coated-conductor films\(^{10}\) and tapes.\(^{8}\) Additionally, the constriction may provide a model for faceted grain boundaries and other periodic systems, as shown in Fig. 5(b). Along the dashed lines, current flows perpendicular to the row of defects, forming lines of constant \( \psi \). Each unit cell of the periodic structure is therefore equivalent to the constriction geometry in Fig. 5(a).

The physical representation of the constriction is mapped onto three regions in the hodograph plane, as shown in Fig. 6. This problem differs from the bridge and current-lead geometries considered in the previous sections, in that in addition to the Dirichlet boundary conditions on the lines \( AB, BC, \) and \( DE \), it contains the Neumann boundary condition \( \partial_\rho \psi = 0 \) on the line \( CD \). Consequently, the full matrix inversion procedure must be applied, which enables us to obtain solutions described in detail in Appendix D. The technique presented in Appendix D is most efficient in the case of defects with \( a > d/n \), for which the effect of the restrictive sample geometry qualitatively changes the nonlinear resistive behavior of the film, as discussed in Sec. II. For very small defects \( a \leq d/n \), the length of the flux jet in Fig. 2 is shorter than the film thickness \( d \), which corresponds to a defect in an infinite superconductor considered in detail in Refs. 22 and 23.

Streamline solutions for the constriction are shown in Fig. 15. Because of the reflection symmetry, only the top half of the solutions are pictured. These exhibit the orientational flow domains (both point-source-like and uniform) and current domain walls, which are most evident when \( n \gg 1 \) and \( (d - a)/d < 1 \). In the aperture limit shown in Fig. 15(a), the current-flow domain bounded under the dot-dashed \( d \) line displays the asymptotic point-source-like behavior. For a confined aperture, in which \( d \) is kept finite, streamlines bend at the current domain walls, attaining a nearly uniform flow state above the characteristic scale \( W \), which describes the size of the current disturbance. For the case \( n \rightarrow \infty \) (dashed lines), current flow is identically uniform in the region \( y \gg W \). The second limiting regime, \( a \ll d \) shown in Fig. 15(b), corresponds to a small planar defect in a film. In this case, current domain walls are still apparent, although the intermediate behavior is no longer point-source-like. [The point-source term is still present in hodograph region (2), Eq. (D5), but does not dominate.]

The spatial distribution of the electric field in a film with a planar defect is quite remarkable, as shown in Fig. 16. The electric-field disturbance is not confined to the vicinity of the crack, but extends all the way across the film in the direction transverse to current flow. A singularity in \( E(x,y) \) occurs at the end of the crack (singular point). Long-range perturbations in \( E(x,y) \) occur, even for small defects, producing nar-
row flux jets (domains of strong electric field), as discussed in Sec. II. In addition, an extensive stagnation region occurs near the point where the crack touches the film edge, where the electric field is exponentially small. The two characteristic electric-field scales shown in Fig. 16 are the asymptotic electric field $E_0$ far from the defect and the peak electric field $E_m$ along the sample edge, opposite to the defect (peak point). The disturbance in $E(x,y)$ has a well-defined width $W_\infty$ in the longitudinal direction, beyond which perturbations decay exponentially. One feature of the current-flow solution not pictured in Fig. 16 includes a delta-function spike in the electric field across the insulating planar defect, reflecting a finite voltage drop between its two sides.

We now investigate these features of the electric-field solution in detail. First we consider the solution along the line $y=0$ between the singular and peak points. Along this symmetry line, the flow angle is given by $\theta = \pi/2$, with $J \gg J_m$. The current-density distribution is then obtained from Eq. (D6):

$$x = a - d + \frac{dJ_0}{J_m} \sum_{m=-\infty}^{\infty} \frac{(-1)^m D_{2m}^{(2)} \tau_{2m-1}^{-1}}{\tau_{2m-1}^{2m-1}} \left( \frac{J}{J_m} \right)^{\tau_{2m-1}^{-1}},$$

(46)

where the coefficients $D_{2m}^{(2)}$ are determined by Eqs. (D9) and (D10) in Appendix D. Near the singular point, the term with $m=1$ in Eq. (46) dominates, giving $J \propto |x + iy + b|^{-1/n}, E \propto 1/|x + iy + b|$ for $n > 1$. Similar singular behavior was also obtained for a planar defect in an infinite superconductor.\textsuperscript{22,23}

The peak values $J_m$ and $E_m = E_0(J_m/J_0)^n$ are important parameters that quantify the strength of the disturbance in $E(x,y)$ produced by the defect. To a first approximation, we find that $J_0/J_m \approx 1-a/d$. By current conservation, this relation becomes exact when $n \rightarrow \infty$, since the current density along the aperture line becomes uniform. For finite $n$, the exact result for $J_0/J_m$ is given in Eq. (D11). The summation of Eq. (D11) enables us to obtain corrections $-1/n$ to the ratio $r = J_0/J_m \approx 1 - a/d$. The following formulas give excellent agreement with the exact summation when $n \gg 1$:

$$\frac{J_m}{J_0} \approx \frac{d}{d-a} \left( 1 - \frac{1}{2n} \right),$$

(47)

$$\frac{E_m}{E_0} \approx \frac{1}{\sqrt{e}} \left( \frac{d}{d-a} \right)^n.$$

(48)

Next we consider the line $x=0$ along the edge of the film, opposite to the defect. Here, the current-flow angle equals $\theta = \pi/2$, and the current density falls into the range $J_0 < J \approx J_m$. From Eq. (D5) we obtain the distribution $J(y)$ in the form

$$y = \frac{2I}{\pi J} - d \sum_{m=1}^{\infty} (-1)^m \frac{2m D_{2m}^{(2)} / J}{\tau_{2m}^{2m-1}} \left( \frac{J}{J_0} \right)^{\tau_{2m}^{-1}}$$

$$- d \sum_{m=1}^{\infty} (-1)^m \frac{2m C_{2m}^{(2)} / J}{\tau_{2m}^{2m-1}} \left( \frac{J}{J_m} \right)^{\tau_{2m}^{-1}},$$

(49)

where the coefficients $C_{2m}^{(2)}$ and $D_{2m}^{(2)}$ are given in Appendix D. The electric-field distribution $E(y)$ described by Eq. (49) is shown in Fig. 17. For $n \gg 1$, even small defects can result in a very strong peak in $E(y)$ over the background electric field $E_0$ far away from the defect.

From Eq. (49), we can obtain the following simple asymptotic expressions for $E(y)$, which reveal characteristic amplitude and length scale of the peak. In the region near the peak $J(y) \approx J_m$, the first sum in Eq. (49) can be neglected, since the factors $(J/J_0)^{\tau_{2m}^{-1}} \sim (J_0/J)^n$ are exponentially small. In the second sum we can take large-$n$ asymptotics of $\tau_{2m}^{\pm}$ and $C_{2m}^{(2)} \sim 2(-1)^m / n! \pi n$ and obtain an excellent approximation for $J(y)$ using Eq. (D20):
The peak portion of the distribution Gaussian shape, with a width $\ln b$ into Eq. (51), behavior of $b$ this equation over $\ln b$ into Eq. (1), again of the order of $L_1$. The second length scale $W_n$ is defined in Eq. (D23), which in the limit $n \rightarrow \infty$ gives

$$W_n = \frac{2d}{\pi} - \frac{4d}{\pi} \sum_{m=1}^{\infty} \frac{(1-a/d)^{4m^2}}{4m^2 - 1}. \quad (53)$$

For large defects $a \ll d$, this rapidly converging sum can be truncated at one or two terms, giving $W_n \approx (2d/\pi) \ln^{1/2} [d/(d - a)]$, thus

$$W_n \approx 2\sqrt{ad/\pi}, \quad a \ll d. \quad (54)$$

As above, $W_n$ quantifies the width of the region (marked in gray in Fig. 15) in which the electric field rapidly converges to its asymptotic value $E_0$. For $n \rightarrow \infty$, the exponentially small tail described by Eq. (54) vanishes, and the electric-field distribution becomes uniform $E = E_0 \hat{y}$ for $x \geq W_n$. The uniform behavior extends across the entire channel. For finite $n$, the electric field is not identically uniform. However, $W_n$ generally marks the onset of exponential decay for all $n$. Since even weak disturbances $\delta E$ can noticeably turn the vector $\mathbf{J}(x,y)$, the length $W_n$ also quantifies the width of orientational current-flow domain that corresponds to the gray area in Fig. 15. As discussed in Sec. II, the length $W_n$ is rather different from the narrow region of exponentially strong electric field in the flux jet, whose width $l_0$ vanishes as $n \rightarrow \infty$. If the channel width $d$ increases to infinity, keeping the defect size $a$ fixed, we see that $W_n \rightarrow \infty$, while $E(y)$ decays on the length $l_0 \sim \pi a$, as shown in Fig. 3. Finally, we consider the line $x = 0$, describing the edge of the film adjacent to the crack. The current-flow angle is again given by $\theta = \pi / 2$, but the current density falls into the range $0 \leq J \leq J_0$. From Eq. (D4) we obtain the appropriate distribution

$$y = -\sum_{m=1}^{\infty} \frac{2md(-1)^m C_{2m}^{(1)} \left( \frac{J}{J_0} \right)^{\tau_{2m}^{-1}}}{\tau_{2m}^{-2m - 1}}. \quad (55)$$

Results for the electric field are shown in Fig. 18, as calculated from Eq. (55). The predominant feature in this figure is the strong depression of $E(y)$ over an extended region near the stagnation point $(x = -d, y = 0)$, which increases in width for increasing $n$ or defect length $a$. Some distance from the stagnation point, the distribution $E(y)$ levels off and converges to the asymptotic value $E = E_0$. Approaching the stagnation point, the lowest-order term in Eq. (55) dominates the sum, giving $E \sim |x + iy| d^{m(\tau_{2m}^{-1} - 1)}$. This power-law behavior is identical to the stagnation point for a planar defect in an
infinite film.\textsuperscript{22,23} When \(n = 1\), we find \(E \propto J \propto |x + iy + d|\), while for \(n \gg 1\), we obtain \(J \propto |x + iy + d|^{1/3}\) and \(E \propto |x + iy + d|^n\).

For \(n \to \infty\) the electric-field distribution becomes singular, approaching the shape of a step function, as shown with dashed lines in Fig. 18. We can determine the location of the step by considering the \(n \gg 1\) limit of Eq. (55). Then Eq. (D7) gives \(C_{2n} = 2(-1)^n(\sqrt{m^2 - 1})/\pi m\) and we make the substitution \(J = J_0\) as appropriate for the top edge of the step. Using \(\Sigma_{n=1}^{\infty} 1/(4m^2 - 1) = \frac{1}{2}\), we obtain \(y = W_\infty\) for the step edge, with \(W_\infty\) defined in Eq. (53). As \(n \to \infty\), the current density is thus uniform for \(y \geq W_\infty\), with \(J = J_1\). For finite \(n\), the current distribution is not identically uniform for \(y \geq W_\infty\), but we observe an exponential convergence of \(J\) and \(E\) to their asymptotic values, similar to Eq. (52).

### VIII. ELECTRIC-FIELD DOMAINS AND CURRENT DOMAIN WALLS

#### A. Flux jets

The results of the previous section show that the electric-field disturbance produced by a small planar defect in a film is localized mainly in a narrow region elongated transverse to current flow. As discussed in Sec. II, this domain of strongly enhanced electric field corresponds to the magnetic-flux jet that connects the end of the defect and the opposite side of the sample surface or other defects (see Fig. 2). The spatial distribution of the electric fields and thus the vortex velocities \(\mathbf{v} = -[\mathbf{E} \times \mathbf{z}]/B_\theta\) is shown in Figs. 16 and 17. Here the electric-field distribution \(E(y)\) along the sample surface, opposite to the defect, is described by the simple Gaussian formula (51).

The nonlinear current flows exhibit the essential difference between the spatial scales of the electric field and current disturbances. While the width of the orientational current-flow domains in restricted geometries is generally of the order of the sample width \(d\), the width of the flux jet is much smaller \(l \sim d/\sqrt{n}\). As the exponent \(n\) increases, the flux jets shrink, while the orientational current-flow domains remain practically unchanged. For arrays of planar defects, such as grain-boundary networks in polycrystals or faceted grain boundaries (see Fig. 2), the flux jets provide a percolative path for continuous magnetic-flux motion. The geometry of this path determines the global critical current and \(E-J\) characteristics of HTS polycrystals, as demonstrated in Sec. IX.

#### B. Orientational current-flow domains

Orientational current-flow domains characteristic of restricted geometries with \(n \gg 1\) are separated by narrow current domain walls where the vector \(\mathbf{J}(x,y)\) sharply changes direction. These domain walls are reminiscent of the \(d\) lines of the Bean model, but they have an internal structure and varying width, which depends both on the exponent \(n\) and on the geometry of current flow.\textsuperscript{22,23} The Bean critical-state model provides the simplest phenomenological description of domain formation, but its multiple solutions are determined by magnetic prehistory (initial conditions). This is demonstrated in Figs. 19(a) and 19(b), where two of an infinite number of possible solutions are pictured.

We now explore current flow away from the domain walls. It is useful to compare solutions for the elbow geometry for finite and infinite \(n\), as described in the Appendix C. An important difference emerges between the two cases.
Whereas the $n \gg 1$ solution describes current flow in the corner region and the two semi-infinite current leads, the infinite $n$ solution describes current flow only in the corner region. For the symmetric elbow, the $n \to \infty$ solution covers a region with the shape of a square, as shown in Fig. 19(c). For the asymmetric elbow, the corresponding solution region is the rectangle with the dimensions $W_\infty \times d$ [Fig. 12(b)]. Outside these shaded regions, the current flowing through the channels is uniform. The width of the current disturbance $W_\infty$ is given by Eq. (42), from which it follows that $W_\infty = b = d$ for the symmetric elbow. For the asymmetric elbow, $W_\infty$ depends on the ratio $b/d$. For instance, if $b/d \to 0$, then Eq. (42) yields $W_\infty = 2d/\pi$.

The location of the current domain walls depends on the geometry, and except for symmetric cases, the boundaries between the orientational flow domains are curved. For instance, for the symmetric elbow, the domain wall is straight and runs along the diagonal, however for the asymmetric elbow, the domain wall becomes curved. For $r \ll 1$, we can qualitatively estimate the shape of the domain wall from the following consideration. We note from Fig. 12(b) that the current flow on either side of the domain wall corresponds asymptotically to either uniform flow (left-hand side), or point-source flow (right-hand side). The boundary between these domains can therefore be identified by matching the two asymptotic solutions. Plugging the relation $y/x = \tan \theta$ from Eq. (27) into Eq. (26), we obtain the point-source stream function in physical coordinates

$$\frac{\psi}{I} = -1 + 2 \left[ \frac{\pi}{2} \arctan \left( \frac{y}{x} + \pi \right) \right].$$

The corresponding relation for uniform flow along the horizontal channel of the elbow is

$$\psi/I = 1 - y/d.$$  (57)

Equating these two results, we obtain the following formula for the position of the domain wall:

$$x = -\frac{y}{\tan(\pi y/2d)}.$$  (58)

This curve is plotted in Fig. 12(b). Note that the $d$ line intersects the $x$ axis at the point $x/d = -2/\pi$, as consistent with Eq. (42) for $W_\infty$.

Likewise, we can estimate the location of the current domain wall for the constriction geometry in Fig. 15(a). Matching the stream functions for the uniform flow and a point source located in the aperture [point $D$ in Fig. 5(a)], we obtain

$$y = \frac{x}{\tan(\pi x/2d)}.$$  (59)

A more general and accurate estimate is obtained by identifying singularities in the streamline curvature, as described below.

C. Current domain walls and $d$ lines

In this section we consider the detailed structure of the current domain wall for the simple symmetric elbow geometry with $b = d$. Figures 19(a) and 19(b) show two rather different solutions for the current streamlines, as predicted by the Bean model. One describes a sharp kink, or $d$ line, in the current flow along the elbow diagonal. The second solution predicts smooth circular streamlines around the corner, with no trace of a $d$ line. Using the hodograph technique, we obtain a unique solution for the current streamlines, as shown in Fig. 19(c). The streamlines exhibit features reminiscent of both the Bean-model solutions, however the magnitude of the current-flow density $J(x,y)$ varies even when $n \to \infty$.

Near the inner corner of the elbow ($z = -b$), the asymptotic solution corresponds to the isolated 270° corner that was studied in Ref. 23. Very near the corner, the streamlines appear circular, similar to Fig. 19(b). In the opposite corner ($z = id$), the asymptotic behavior is that of an isolated 90° corner, also studied in Ref. 23. In that case, the streamlines remained smooth, however a kink, or $d$ line, arose in the gradient of the current-flow angle $\nabla \theta$ as $n \to \infty$.

We investigate the emergence of the $d$ line in the elbow as $n \to \infty$, considering a new set of coordinates in the physical plane $l_\parallel$ and $l_\perp$:

$$l_\parallel = \frac{1}{\sqrt{2}} (x + y + b), \quad l_\perp = \frac{1}{\sqrt{2}} (y - x - b),$$  (60)

as shown in the inset of Fig. 20(b). Figure 20(a) shows the spatial distribution of the orientation of $\mathbf{J}(x,y)$ in the domain wall described by the angle $\delta(l_\perp) = \theta(x,y) - 3\pi/4$ between the local vector $\mathbf{J}(x,y)$ and the direction of $l_\perp$. The function $\delta(l_\perp)$ exhibits a kink at $l_\perp = 0$, which becomes sharper as $n$ increases. However $\delta(l_\perp)$ remains continuous for any $n$, as opposed to the jumpwise behavior $\delta(l_\perp) = -\pi \sgn(l_\perp)/4$ of the Bean model. To quantify the slope of $\delta(l_\perp)$ at $l_\perp = 0$, we calculated the derivative $\partial \delta/l_{\perp}
= 0$ using the solutions obtained in Appendix C. The results are shown in Fig. 20(b). For increasing $n$, the flow-angle derivative increases, diverging along the entire diagonal as $n \to \infty$. Thus, for $n = \infty$, the flow-angle derivative contains a singularity along the curve $l_\perp = 0$, which defines the center of the current domain wall. We point out that $\partial \delta/l_{\perp} = 0$ plotted in Fig. 20(b) is also singular at the points $l_\parallel = 0$ and $\sqrt{2}b$, even for finite $n$. This simply reflects the sharp turn of the current at the corner as it flows tangentially along the sample boundary.

The nature of the singularity can be investigated in further detail. Very near the $d$ line, we have $\theta = 3\pi/4 + \delta$ with $|\delta| \ll 1$. Expanding the formula for $l_\perp(J,\theta)$, Eq. (C12), in small powers of $\delta$, we obtain

$$l_\perp = \nu b \left( \frac{3}{n} \delta + \delta^3 \right),$$  (61)

where $\nu$ involves an infinite sum that depends only on $l_\parallel$ (or $J$). This same expression was also found for the 270° corner. In that case, $\nu$ was given by the term with $m = 1$ in Eq. (C12).
The structure of the current-domain wall for the symmetric elbow geometry. (a) Current-flow angle ($\theta$) vs transverse coordinate $l_\perp$, calculated along the line $l_\parallel=d$. Curves correspond to $n=30, 100, 300$ (left to right). The dashed line corresponds to the Bean model. Inset shows the flow-angle derivative $\partial \theta / \partial l_\perp$ vs $l_\perp$. (b) Flow-angle derivative vs the coordinate $l_\parallel$ along the diagonal at $l_\parallel=0$. Curves correspond to $n=1, 5, 10, 20$ (bottom to top).

From Eq. (61), we obtain that $\theta(l_\perp)$ near the center of the current domain wall for $n \rightarrow \infty$ is given by

$$\theta \approx \frac{3\pi}{4} + \alpha \left( \frac{l_\perp}{b} \right)^{1/3},$$

where $\alpha$ depends only upon $l_\parallel$, as shown in Fig. 20(b). The singularity in the flow-angle derivative at $l_\parallel=0$ therefore takes the form $\partial \theta / \partial l_\perp \propto l_\perp^{-1/3}$, analogous to the 270° corner. For finite $n$, $\theta(l_\perp)$ becomes a smooth function of $l_\perp$ with no singularities in the derivatives and a finite slope $\sim n/b$ at $l_\parallel=0$.

Now we consider the width of the current domain wall where the flow angle $\theta(x,y)=3\pi/4$ varies rapidly. As follows from Eqs. (61) and (62), the power-law dependence of $\theta(x,y)$ does not have any intrinsic scales that would unambiguously quantify the width of the current domain wall. The domain wall can therefore be defined as a region confined within a range of flow angles $|\delta(x,y)|<\delta_0$, say $5\pi/8<\theta<7\pi/8$. The so-defined region is shown in Fig. 19(c) for several values of $n$. We see that the current domain wall has a varying finite width even in the critical-state limit ($n \rightarrow \infty$). The width depends on a conditional criterion $\delta_0$. Thus the structure of the current domain wall in confined geometries is very different from the phenomenological $d$ lines of the Bean model. This result reinforces the similar conclusion obtained in our previous work for nonlinear current flow around semi-infinite corners and planar defects in infinite superconductors.

### IX. Global Current-Voltage Characteristics and Local Dissipation on Planar Defects

The strong electric-field enhancement in the flux jets around planar defects results in local voltage steps, which can essentially affect macroscopic transport properties of superconductors. As an example we consider a long section of film containing a planar defect. The excess voltage $\Delta V = \int [E-E_0] \cdot d\mathbf{l}$ associated with the defect can be written in the form

$$\Delta V = 2 \int_{E_0}^{E_m} (E-E_0) \frac{\partial y}{\partial E} dE,$$

where $y$ is given in Eq. (49). Substitution of Eq. (49) into Eq. (63) and integration over $J$ gives a cumbersome formula for $\Delta V$, which we used for numerical calculations of $\Delta V$ shown in Fig. 21 (solid lines). The disturbance is most pronounced for large defects and large $n$. For a superconductor with $n \approx 1$, a single planar defect (crack or high-angle grain boundary) may radically affect macroscopic transport properties, because of the significant enhancement of the electric field in the flux jet of width $l_0 \sim d/\sqrt{n}$. A simple analytical formula for $\Delta V$ can be obtained integrating the Gaussian peak (51) of the electric field near defect. This yields $\Delta V = \sqrt{\pi l_0 E_m}$, thus

$$\Delta V = \frac{2dE_0}{\sqrt{n}e} \left( \frac{d}{d-a} \right)^{n-1}.$$  

This formula gives an excellent approximation for $\Delta V$. For $n \gg 1$, the excess voltage $\Delta V$ rapidly increases with the defect size $a$.

Distribution of the electric field $E(y)$ along the film edge adjacent to the crack is shown in Fig. 18. There is a very
strong suppression of $E(y)$ in the stagnation region of width $W_y$, where the electric field is exponentially small as compared to $E_0$. In addition, there is a peak in $E(y)$ on the crack, which corresponds to the delta function in the electric field $E(y) = V_c \delta(y)$. The value of $V_c$ can be estimated from the condition $\int E \cdot dl = 0$ along the rectangular contour of width $\gg W_y$ and height $d$, which goes along both film edges. This condition gives $\Delta V = V_c = 2W_y E_0$ if we neglect the exponentially small $E(y)$ in the stagnation region. Hence,

$$V_c = \Delta V + 2W_y E_0.$$  

(65)

Results of calculations of $V_c$ are shown as dashed lines in Fig. 21.

Now we consider the influence of an array of planar defects on the global critical currents (Fig. 22). If the mean spacing $L_i$ between defects is much larger than the width of the current disturbance $W_y \sim \sqrt{a_i d}$, the local voltage peaks do not overlap, thus the total voltage $V$ on the sample is given by

$$V = E_0 \left[ L + \frac{2d}{\sqrt{a_i d}} \sum_i \left( \frac{d}{d-a_i} \right)^{n-1} \right],$$  

(66)

where $E_0$ is the uniform electric field between the defects (white regions in Fig. 22), and the summation goes over all defects of length $a_i$ sticking out from the film edges. If defects are in the center of the film, as shown in Fig. 22, the factor $2d$ in front of the sum should be replaced by $d$. Notice that for $n \gg 1$, the total voltage $V$ may be dominated by a few large defects, so $V$ is very sensitive to the large-$a$ tail of the defect size distribution function $P(a)$.

As an illustration, we consider an array of edge planar defects of the same length $a$, perpendicular to the current flow. Then Eq. (66) gives the following expression for the averaged electric field $\bar{E} = V/L$:

$$\frac{\bar{E}}{E_0} = 1 + \frac{2d}{L \sqrt{a_i d}} \left( \frac{d}{d-a} \right)^{n-1},$$  

(67)

where $V$ is the total voltage on the sample of length $L$, and $L_i$ is the spacing between the defects. Taking $L_i = 2d$ and $n = 20$, we obtain that the last term on the right-hand side of Eq. (67) equals $9.4$ for $a = 0.2d$ and $119$ for $a = 0.3d$. Thus, for $n \gg 1$, even a sparse array of small defects ($a \ll d$) can dominate the global electric field $\bar{E}$.

Now we calculate the global voltage-current characteristic $V(I)$. Because of the linear relation (67) between the mean electric field $\bar{E}$ and the uniform electric field $E_0$ between defects (white regions in Fig. 22), the global $V(I)$ remains the power-law dependence $V = V_c (I/I_c)^n$ with the same exponent $n$ as for a defect-free superconductor. The fact that macroscopic defects do not change the functional dependence of $V(I)$ is a unique scaling feature of the power-law $E$-$J$ characteristics. At the same time, the global critical current $I_c$ can be substantially reduced due to blockage of current flow by a network of planar defects and the corresponding increase of the mean electric field $\bar{E} = E_c (I/I_c)^n$.

To compare $I_c$ and $\bar{I}_c$, we define the global $\bar{I}_c$ at the same mean electric field $\bar{E} = E_c$ as $I_c$ for a defect-free film. To express $\bar{I}_c$ in terms of $I_c$, we note that if $I = I_c$, then the electric field in the white regions of uniform current flow between the defects in Fig. 22 equals $E_c$, since $E = E_c (I/I_c)^n$. Hence it follows that $\bar{E} = E_c (I_c/I_c)^n$, where $E$ is given by Eq. (67) with $E_0 = E_c$. Thus, we obtain the global critical current $I_c = (E_c L V)^{1/n} \bar{I}_c$ in the form

$$\bar{I}_c = \left[ 1 + \frac{2d}{L \sqrt{a_i d}} \sum_i \left( \frac{d}{d-a_i} \right)^{n-1} \right]^{-1/n} I_c.$$  

(68)

This expression becomes more transparent for an array of defects of the same length $a$ at $n \gg 1$ if the first term in the
square brackets in Eq. (68) can be neglected. Defining the global current density \( \overline{J}_c = \overline{I}_c / A \), where \( A \) is the cross section of the sample, we obtain

\[
\overline{J}_c = J_c \left( 1 - \frac{a}{d} \right) \left[ \frac{L_i \overline{\alpha}}{2(d-a)} \right]^{1/n}.
\]

(69)

Here the factor \( 1 - a/d \) results from the reduction of current-carrying cross section by the defect, and the factor in the square brackets accounts for a weak effect of \( \overline{E} \) and the distribution of current flow on \( \overline{J}_c \). Taking, for example, \( a = 0.3d \), \( L_i = 5d \), and \( n = 20 \), we obtain \( \overline{J}_c = 0.82J_c \).

The influence of defects on global dissipation power \( \overline{Q} = IV \) in superconductors can be obtained from Eqs. (66)–(68). We consider two regimes in which defects affect \( \overline{Q} \) in different ways. The first regime corresponds to a fixed current mode \( I_0 = \text{const} \), characteristic of transport measurements for which

\[
\overline{Q} = I_0 E_0 L \left[ 1 + \frac{2d}{L \sqrt{\alpha n}} \sum_i \left( \frac{d}{d-a_i} \right)^{n-1} \right].
\]

(70)

In this case defects can considerably increase \( \overline{Q} \) as compared to a uniform superconductor for which \( \overline{Q} = I_0 E_0 L \). For instance, even for a sparse array of comparatively small defects with \( L_i = 5d \) and \( n = 30 \), the enhancement factor in the square brackets is about 30 for \( a = 0.2d \) and \( 1.4 \times 10^3 \) for \( a = 0.3d \).

In a regime of fixed averaged electric field (or total voltage) \( \overline{E} = E_c \), the global dissipation power \( \overline{Q} = LE_c I_0 \) equals

\[
\overline{Q} = LE_c \overline{I}_c,
\]

(71)

since the global critical current \( \overline{I}_c \) is defined at the same averaged field \( E_c \), as \( I_c \) for a defect-free superconductor. Because of the current blockage by defects, the global \( \overline{I}_c \) is smaller than \( I_c \), so in the fixed voltage mode, the total dissipation power \( \overline{Q} \) is decreased by defects. The above results can be important for calculation of ac losses in HTS (see, e.g., Refs. 36 and 37).

**X. DISCUSSION**

In this work, we have applied the hodograph technique to uncover features of transport current flow in superconductors for basic restricted geometries shown in Fig. 2. The obtained closed-form solutions enabled us to take into account the strongly nonlinear \( E-J \) relation in superconductors and calculate such characteristics as distributions of the electric field and dissipation, which principally cannot be addressed in the Bean model. The hodograph solutions for the current distributions share certain similar features that appear intermediate between the extreme cases \( n = 1 \) and \( n \to \infty \). For instance, we find that solutions describing the current streamlines for \( n > 3 \) are rather close to those for \( n \to \infty \). This fact generally makes it difficult to reveal the finite \( n \) effects from magneto-optical images of HTS.

However, although the current streamlines are weakly dependent on \( n \) for \( n > 1 \), the electric-field distributions remain very sensitive to \( n \). Indeed, from \( E = E_n(J/J_c)^n \), it follows that for \( n > 1 \), even weak variation of the current-flow patterns with \( n \) essentially changes the corresponding \( E(x,y) \) distribution. For example, the width of the flux jets decreases as \( 1/\sqrt{n} \), while the magnitude \( E_{ac} \) in the jet increases with \( n \), as \( [d/(d-a)]^n \) [see Eqs. (48) and (51)]. We therefore gain no information about the electric-field distributions from the \( n \to \infty \) solution for \( J(x,y) \), which is close to static current-flow patterns reconstructed from magneto-optical images of HTS. The only consistent way of calculation of \( E(x,y) \) and thus the steady-state dissipation power \( Q(x,y) = JE \) is to solve the Maxwell Eq. (1) with the account of the nonlinear \( E(J) \) characteristic.

For the cases shown in Fig. 2, the strong nonlinearity of \( E(J) \) gives rise to current-flow patterns that exhibit orientational current-flow domains. The domain size \( W_{ac} \) is determined by the particular sample geometry and is independent of \( n \) for \( (n \geq 1) \). For instance, \( W_{ac} \sim d \) for an asymmetric elbow or \( W \sim \sqrt{ad} \) for a planar defect in a film. The flow patterns in these orientational domains have continuously varying current density [such as the point-source domain in Fig. 12(b)] and thus cannot be described by the Bean model, which requires constant \( J = J_c \). The orientational flow domains are separated by current domain walls reminiscent of the \( d \) lines, yet having quite a different structure, as discussed in the previous sections.

For finite \( n \) values, the current-domain walls described by exact hodographic solutions have neither discontinuities nor singularities in \( J(x,y) \) and \( E(x,y) \). Even in the limit \( n \to \infty \), the current-flow angle \( \theta(x,y) \) remains continuous, but its gradient has a singularity, as given by Eq. (62). Because of a progressive broadening of the domain wall with the distance from an isolated inner corner or from the crack in the infinite film, it becomes more poorly defined far from its source.

The hodograph solutions reveal flux jets of strongly enhanced electric fields and dissipation in restricted geometries. These jets cause substantial current-blocking effects, even if defects occupy a small fraction of the geometrical cross section of a superconductor. Unlike the orientational current-flow domains, the flux jets shrink as \( n \) increases and turn into singular lines in the limit \( n \to \infty \). Because of the exponential increase of \( E_m \) in the jets, the global \( \overline{J} \)-characteristics in the flux creep region \( \overline{E} \ll E_c \) become very sensitive to macroscopic planar defects. Thus, the global \( \overline{J} \)-relation at \( \overline{E} \ll E_c \) can be dominated by a sparse array of defects rather than “bulk” properties of vortex dynamics and pinning on mesoscopic scales of order of the Larkin length \( L_c \).

The local electric-field enhancement and singularities in \( E(x,y) \) near the edges of planar defects can induce flow flux flow \( (E > E_c) \) within the flux jets, even if the electric field \( E_0 \) far from the defect is much smaller than the flow flux threshold \( E_c \) (see Fig. 1). In this case the power-law approximation for \( E(J) \) becomes inadequate. In Ref. 23, a realistic \( E-J \) relation was used to investigate current and flux flow near a semi-infinite crack. Although no similar analysis has been performed in the present work, we can make general comments...
about flux flow. First, we note that a flux-flow region is always induced near the end of a planar defect or an outer corner. For a planar defect, the size of the flux-flow domain is of the order \( R \sim a \sqrt{nE_0/\xi} \). For a planar defect in a film, the overall electric field \( E_m \) in the flux jet given by Eq. (49) exceeds the flux-flow threshold \( \xi \) if

\[
a > a_c \sim d \left( 1 - \left( \frac{E_0}{E_c} \right)^{\ln(2)} \right). \tag{72}
\]

For \( n = 30 \) and \( E_0 = 10^{-3}E_c \), Eq. (72) gives \( a_c \sim 0.2d \). By its nature, the flux-flow domain is highly resistive, compared to nearby superconducting regions. The most serious consequence of flux flow in vortex jets is therefore current blockage, so cracks in thin films and coated conductors can strongly degrade global-transport properties. If the flux-flow domain occupies the entire cross section, a local hot spot and strongly degrade global-transport properties. If the flux-flow domain occupies the entire cross section, a local hot spot and perhaps thermal instabilities can occur. An analysis of these effects will be given elsewhere.

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**APPENDIX A: CONVERGENCE TECHNIQUES**

Here we summarize the methods that we used to improve the convergence of the hodograph series in Eqs. (20)–(22). The expansion coefficients \( C_m \) and \( D_m \) typically converge as \( \mathcal{O}(1/m) \), so that summations may require up to 1000 terms to achieve high accuracy. To improve the situation, we added and subtracted a known function to Eqs. (20) and (22), whose series expansion has the same asymptotic behavior at \( m \gg 1 \) as \( C_m \) and \( D_m \). In this way, convergence is improved from \( \mathcal{O}(1/m) \) to \( \mathcal{O}(1/m^2) \), reducing computation times by a factor of 10 or more.

Method 1. For certain geometries, such as the bridge or elbow, the coefficients \( C_m \) and \( D_m \) can be computed exactly, so that the large-\( m \) behavior of \( C_m \) and \( D_m \) is known. For \( m \gg 1 \), we also have \( \tau_m \sim \sqrt{nm + (1 - n)/2} \), thus

\[
\frac{\psi}{I} = \sum_{m=1}^{\infty} \left[ C_m \left( \frac{J}{J_0} \right)^{\tau_m} - \chi_m \left( \frac{J}{J_0} \right)^{-\tau_m} \right] \sin(m \theta) + \frac{J}{J_0}^{(1-n)/2} \int_{-\theta}^{\theta} \left[ \frac{J}{J_0}^{\tau} \right] \sin \theta. \tag{A1}
\]

\[
f(x, \theta) = \sum_{m=1}^{\infty} \chi_m x^m \sin(m \theta). \tag{A2}
\]

As \( N \to \infty \), Eq. (A1) becomes an identity. The convergence of the series is now improved by a factor of \( 1/N \).

Method 2. For certain geometries, such as a planar defect in a film, explicit expressions for \( C_m \) and \( D_m \) are unknown. However it is usually possible to calculate those coefficients using known and much simpler solutions for Ohmic conductors of the same geometry:

\[
\frac{\psi}{I} = -1 + \frac{2}{\pi} \theta + \sum_{m=1}^{\infty} C_{2m} \left( \frac{J}{J_0} \right)^{\tau_{2m}} \sin(2m \theta) \quad \text{[region (1)]}, \tag{B1}
\]

\[
\psi = \sum_{m=1}^{\infty} \left( C_m \left( \frac{J}{J_0} \right)^{\tau_m} - \chi_m \left( \frac{J}{J_0} \right)^{-\tau_m} \right) \sin(m \theta), \tag{A3}
\]

where \( \tau_m = m \), as appropriate for \( n = 1 \), and the \( C_m \) series is omitted for simplicity.

We use the identity \( D_m = \gamma_D d_m \), which holds to leading order for large \( m \). Here, the constant \( \gamma_D \) depends on \( n \) (but not \( m \)), while \( d_m \) depends on \( m \) (but not \( n \)). We note that \( K_{m,m'} \) and \( G_m \) in Eq. (24) are known exactly, as are the corresponding \( m = 1 \) quantities \( k_{m,m'} \) and \( g_m \). When \( m \to \infty \), we find that \( K_{m,m'} \to \gamma_k \delta_{m,m'} \), where \( \gamma_k \) depends only on \( n \) while \( K_{m,m'} \) depends only on \( m \) and \( m' \). The add/subtract trick now gives

\[
\sum_{m'=1}^{N} \left( K_{m,m'} D_{m'} - \gamma_k \gamma_D k_{m,m'} d_{m'} \right) = G_m - \gamma_k \gamma_D g_m. \tag{A4}
\]

In the limit \( N \to \infty \), this equation becomes exact. However the expression inside square brackets is now rapidly convergent, allowing the summation to be truncated at a modest \( N \) value. For typical situations, we use \( N = 50 – 100 \).

The \( N \times N \) matrix inversion is now manageable. From Eq. (A4), we have

\[
D_m = \gamma_D X_m + Y_m, \tag{A5}
\]

\[
X_m = \gamma_k \sum_{m'=1}^{N} K_{m,m'}^{-1} \sum_{m''=1}^{N} k_{m'',m'} d_{m''} - g_{m'} \tag{A6}
\]

\[
Y_m = \sum_{m'=1}^{N} K_{m,m'}^{-1} G_{m'}. \tag{A6}
\]

The parameter \( \gamma_D \) is calculated from Eq. (A5). We assume that the truncation index \( N \) is chosen large enough that coefficient \( D_m \) has converged to its asymptotic form: \( D_N = \gamma_D d_N \). Combining this with Eq. (A5) gives

\[
\gamma_D = Y_N / (d_N - X_N). \tag{A7}
\]

**APPENDIX B: BRIDGE CALCULATIONS**

The line \( J = J_0 \) divides the hodograph representation, Fig. 8(b), into two distinct BC regions. In region (1), \( J \to 0 \) leads to \( D_m (J/J_0)^{\tau_m} \to \infty \). [See Eq. (20).] Since the stream function should be finite, the \( D_m \) coefficients must vanish. Similarly in region (2), \( J \to \infty \) leads to \( C_m (J/J_0)^{\tau_m} \to \infty \), so that the \( C_m \) coefficients must vanish. Using Eq. (20), we obtain the following general expressions for the stream function, which satisfy all external BC:

\[
\frac{\psi}{I} = -1 + \frac{2}{\pi} \theta + \sum_{m=1}^{\infty} C_{2m} \left( \frac{J}{J_0} \right)^{\tau_{2m}} \sin(2m \theta) \quad \text{[region (1)]}, \tag{B1}
\]

\[
\frac{\psi}{I} = -1 + \frac{2}{\pi} \theta + \sum_{m=1}^{\infty} C_{2m} \left( \frac{J}{J_0} \right)^{\tau_{2m}} \sin(2m \theta) \quad \text{[region (2)]}, \tag{B2}
\]
\[
\psi/J = 1 + \sum_{m=1}^{\infty} D_{2m} \left( \frac{J}{J_0} \right)^{\tau_{2m}^+} \sin(2m\theta) \quad \text{[region (2)]}. 
\]  
(B2)

We can immediately obtain expressions for \( z(J, \theta) \) from Eq. (22):

\[
z = e^{i\theta} \frac{2J}{\pi J} - be^{i\theta} \sum_{m=1}^{\infty} \frac{C_{2m}}{\tau_{2m}^+ - 1} \left( \frac{J}{J_0} \right)^{\tau_{2m}^- - 1} \left[ 2m \cos(2m\theta) \right] - i\tau_{2m}^+ \sin(2m\theta) \quad \text{[region (1)]}, 
\]  
(B3)

\[
z = -b - be^{i\theta} \sum_{m=1}^{\infty} \frac{D_{2m}}{\tau_{2m}^- - 1} \left( \frac{J}{J_0} \right)^{\tau_{2m}^- - 1} \left[ 2m \cos(2m\theta) \right] - i\tau_{2m}^+ \sin(2m\theta) \quad \text{[region (2)]}, 
\]  
(B4)

where \( b = I/J_0 \). The integration constants \( F_0 \) in Eq. (22) have been determined as follows. In region (1), when \( J \to 0 \), the dominant term in Eq. (B3) is the first term on the right-hand side, which describes point-source behavior. By considering both halves of the bridge geometry in Fig. 8(a), we see that the effective point source should be located at the center of the channel opening at \( x = y = 0 \). This corresponds exactly to the point-source configuration of Sec. IV, and by analogy we obtain \( F_0 = 0 \). In region (2), the singularity \( J \to \infty \) occurs at the sharp corner at \( x + iy = -b \), and we obtain \( F_0 = -1 \).

To determine \( C_m \) and \( D_m \), we apply matching conditions [Eq. (23)] along the line \( J = J_0 \). This gives

\[
\frac{2}{\pi} \theta = \sum_{m=1}^{\infty} \left[ D_{2m} - C_{2m} \right] \sin(2m\theta),
\]

\[
0 = \sum_{m=1}^{\infty} \left[ \tau_{2m}^- D_{2m} - \tau_{2m}^+ C_{2m} \right] \sin(2m\theta). 
\]  
(B5)

We obtain \( C_m \) and \( D_m \) using the Fourier analysis by multiplying Eq. (B5) by \( \sin(2m\theta) \) and integrating over \( \theta \) in the interval \( \pi/2 \leq \theta \leq \pi \). Hence,

\[
C_{2m} = (-1)^m \frac{2}{\pi m} \frac{\tau_{2m}^-}{\tau_{2m}^- - \tau_{2m}^+}, \quad D_{2m} = (-1)^m \frac{2}{\pi m} \frac{\tau_{2m}^+}{\tau_{2m}^- - \tau_{2m}^+}. 
\]  
(B6)

The expansions (B1)–(B4) converge slowly when \( J \approx J_0 \). We therefore apply the techniques of Appendix A (method 1). For Eq. (B1), we find

\[
C_{2m} \to \chi_{2m} = \frac{(-1)^m}{\pi m}, \quad f(x, \theta) = -\frac{1}{\pi} \text{Im}[\ln(1 + x^2 e^{2i\theta})].
\]  
(B7)

Equations (B2)–(B4) are treated in the same manner.

We now analyze the above solutions in the limiting case \( n \to \infty \), for which

\[
\tau_m^+ \to m^2, \quad \tau_m^- \to -\infty. 
\]  
(B8)
hodograph analysis, the matching conditions, applied previously along the line $J = J_0$, are now replaced by the Dirichlet condition $\psi = I$.

**APPENDIX C: CURRENT-LEAD CALCULATIONS**

We form a complete solution for each hodograph region, satisfying all the external BC of Fig. 11(b):

$$\psi = \sum_{m=1}^{\infty} C_{2m} [J / J_{1}]^{\tau_{2m}^+} \sin(2m\theta) \quad \text{[region (1)]},$$

$$\psi = -1 + \frac{2}{\pi} \frac{\psi}{\tau_{2m}} \sum_{m=1}^{\infty} D_{2m} [J / J_{1}]^{\tau_{2m}^-} \sin(2m\theta) \quad \text{[region (2)]},$$

$$\psi = \psi_{1}^{\infty} \frac{\psi}{\tau_{2m}^-} \sum_{m=1}^{\infty} D_{2m} [J / J_{1}]^{\tau_{2m}^-} \sin(2m\theta) \quad \text{[region (3)]}.$$

The relation $z = z(J, \theta)$ is obtained from Eq. (22):

$$z = \sqrt{\frac{2i}{\pi J}} \psi_{1}^{\infty} \frac{\psi}{\tau_{2m}^-} \sum_{m=1}^{\infty} D_{2m} [J / J_{1}]^{\tau_{2m}^-} \sin(2m\theta) \quad \text{[region (1)]},$$

$$z = \sqrt{\frac{2i}{\pi J}} \psi_{1}^{\infty} \frac{\psi}{\tau_{2m}^-} \sum_{m=1}^{\infty} D_{2m} [J / J_{1}]^{\tau_{2m}^-} \sin(2m\theta) \quad \text{[region (2)]},$$

$$z = -b - \sqrt{\frac{2i}{\pi J}} \psi_{1}^{\infty} \frac{\psi}{\tau_{2m}^-} \sum_{m=1}^{\infty} D_{2m} [J / J_{1}]^{\tau_{2m}^-} \sin(2m\theta) \quad \text{[region (3)]}.$$

Here the superscript notations refer to a particular hodograph region. Note that the hodograph expansions involving coefficients $C_{2m}$ and $D_{2m}$ are obtained in terms of the dimensionless variables $z / b$ and $J / J_{1}$. However here we make explicit reference only to the current scale $J_{1}$ and the length scale $d$, using the relations $b = r d$ and $J_{2} = J_{1} / r$.

The integration constants $F_0$ of Eq. (22) can be determined according to the following arguments. In region (1), the current vanishes ($J \rightarrow 0$) at the stagnation point $z = i d$, giving $F_0 = i$. In region (2), the current diverges ($J \rightarrow \infty$) at the corner singularity $z = -b$, giving $F_0 = -d$. In region (3), we note the presence of asymptotic point-source flow, corresponding to a nonvanishing $C_0$ term. Analogous to arguments given for the bridge geometry, we infer the location of the effective point source to be the origin $z = 0$, thus $F_0 = 0$.

We must now apply matching conditions, Eq. (23), along two hodograph boundaries $J = J_1$ and $J = J_2$. This produces four equations relating the four coefficient sets $C_{2m}^{(1)}, D_{2m}^{(1)}, C_{2m}^{(2)},$ and $D_{2m}^{(2)}$. Applying a Fourier analysis over the interval $\pi / 2 \leq \theta \leq \pi$ gives

$$\begin{pmatrix}
-1 & 1 & r_{2m}^+ & 0 \\
0 & r_{2m}^- & 1 & -1 \\
-r_{2m}^- & r_{2m}^+ & r_{2m}^+ & 0 \\
0 & r_{2m}^+ & -r_{2m}^- & -r_{2m}^-
\end{pmatrix}
\begin{pmatrix}
C_{2m}^{(1)} \\
D_{2m}^{(1)} \\
C_{2m}^{(2)} \\
D_{2m}^{(2)}
\end{pmatrix}
= \begin{pmatrix}
1 \\
(1)^m \\
0 \\
0
\end{pmatrix}.$$

Thus, we obtain the following solutions:

$$C_{2m}^{(1)} = \frac{2}{\pi m} \frac{\tau_{2m}^+}{(\tau_{2m}^- - \tau_{2m}^+)} [1 - (1)^m r_{2m}^+] \quad \text{[region (1)]},$$

$$D_{2m}^{(1)} = -\frac{2}{\pi m} \frac{\tau_{2m}^+}{(\tau_{2m}^- - \tau_{2m}^+)}.$$

$$C_{2m}^{(2)} = \frac{2}{\pi m} \frac{\tau_{2m}^-}{(\tau_{2m}^- - \tau_{2m}^+)} (1)^m \quad \text{[region (2)]},$$

$$D_{2m}^{(2)} = \frac{2}{\pi m} \frac{\tau_{2m}^+}{(\tau_{2m}^- - \tau_{2m}^+)} [(1)^m - r_{2m}^-].$$

In the limit $n \rightarrow \infty$, the upper hodograph region (3) vanishes. Taking the limit $n \rightarrow \infty$, we obtain

$$C_{2m}^{(1)} = \frac{2}{\pi m} [1 + (1)^m r_{4m}^2], \quad C_{2m}^{(2)} = \frac{2}{\pi m} (1)^m,$$

$$D_{2m}^{(2)} = D_{2m}^{(3)} = 0, \quad r_{4m}^2 = 4m^2.$$

Finally, we give the expression for the symmetric elbow coordinate $l_{\perp}$ defined in Eq. (60):

$$l_{\perp} = \frac{b}{\sqrt{2}} \sum_{m=1}^{\infty} \frac{C_{4m-2}^{(1)}}{\tau_{4m-2}^+} [J / J_{1}]^{\tau_{4m-2}^+} \sin(4m-2 \theta) [\cos(\theta) - \sin(\theta)]$$

$$\times \{ (4m-2 \cos(4m-2 \theta)) [\cos(\theta) - \sin(\theta)]$$

$$+ \tau_{4m-2}^+ \sin(4m-2 \theta) [\cos(\theta)$$

$$+ \sin(\theta)] \} \quad \text{[region (1)].}$$

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APPENDIX D: PLANAR DEFECT IN A FILM

1. Determination of coefficients

A general solution that satisfies all boundary conditions in Fig. 5(a) is given by

$$\psi = 1 + \sum_{m=1}^{\infty} C^{(1)}_{2m} \left( \frac{J}{J_0} \right) \tau^+_m \sin(2m \theta) \quad \text{(region 1)},$$

$$\psi = -1 + \left( \frac{2}{\pi} \right) \theta + \sum_{m=1}^{\infty} D^{(2)}_{2m} \left( \frac{J}{J_0} \right) \tau^-_m + C^{(2)}_{2m} \left( \frac{J}{J_m} \right) \sin(2m \theta) \quad \text{(region 2)},$$

$$\psi = 1 + \sum_{m=1}^{\infty} D^{(3)}_{2m-1} \left( \frac{J}{J_0} \right) \tau^-_{2m-1} \times \sin[(2m-1) \theta] \quad \text{(region 3)},$$

where $J_m$ is defined as the current density at the origin ($z=0$). Note that Eq. (D3) contains a sine term with an odd multiple of $\theta$ in its argument. This new type of term arises from the Neumann BC in hodograph region (3).

The solution for $z(J, \theta)$ is given by

$$z = -d - d e^{i\theta} \sum_{m=1}^{\infty} \frac{C^{(1)}_{2m} \left( \frac{J}{J_0} \right) \tau^+_m}{\tau^+_m - 1} \quad \text{(region 1)},$$

$$z = e^{i\theta} \frac{2J}{\pi J} - d e^{i\theta} \sum_{m=1}^{\infty} \frac{D^{(2)}_{2m} \left( \frac{J}{J_0} \right) \tau^-_m}{\tau^-_m - 1} \quad \text{(region 2)},$$

$$z = -d + a - \tau^-_m \sin(2m \theta) \quad \text{(region 3)},$$

where $r = J_0/J_m$, and the integration constants have been determined analogously to Appendix C.

By applying matching conditions, Eq. (23), along the hodograph boundaries $J = J_0$ and $J_m$, we obtain equations relating the coefficients $C^{(1)}_{2m}$, $D^{(2)}_{2m}$, $C^{(2)}_{2m}$, and $D^{(3)}_{2m-1}$. Using Fourier analysis in the interval $\pi/2 < \theta < \pi$, we reduce the problem to the form of Eq. (24) and express $C^{(1)}_{2m}$ and $C^{(2)}_{2m}$ in terms of the coefficient $D^{(3)}_{2m-1}$:

$$D^{(2)}_{2m} = \frac{2(-1)^m \tau^+_m}{\pi m (\tau^+_m - \tau^-_m)}, \quad C^{(1)}_{2m} = D^{(2)}_{2m} + \tau^+_m C^{(2)}_{2m} - \frac{2(-1)^m}{\pi m},$$

$$C^{(2)}_{2m} = \frac{2(-1)^m}{\pi m} \left( 1 - r^{-\tau^-_m \tau^+_m} \right) \frac{1}{\tau^+_m - \tau^-_m} + \frac{8}{\pi} \sum_{m'=1}^{\infty} \left( \frac{(1-m^m m)}{(2m'-1)^2 - (2m)^2} \right) D^{(3)}_{2m'-1}.$$}

In turn, the coefficient $D^{(3)}_{2m-1}$ is determined by the following matrix inversion procedure:

$$D^{(3)}_{2m-1} = \sum_{m'=1}^{\infty} K^{-1}_{m,m'} G_{m'},$$

$$K_{m,m'} = \frac{(-1)^m (r^{\tau^+_m} - r^{-\tau^-_m})}{(2m'-1)^2 - (2m)^2},$$

$$G_m = \frac{(-1)^m \tau^+_m (r^{-\tau^-_m} - 1)}{(2m)^2}.$$}

Here the parameter $r = J_0/J_m$ is determined from Eq. (D6), using the condition that $J = J_m$ for $z = 0$ and $\theta = \pi/2$. This gives

$$\frac{1}{r} = \frac{d}{d-a} \sum_{m=1}^{\infty} \frac{(-1)^m \tau^-_{2m-1}}{\tau^+_m - \tau^-_m}.$$}

Since $D^{(3)}_{2m-1}$ itself depends on $r$, the quantities $D^{(3)}_{2m-1}$ and $r$ must therefore be determined self-consistently from Eqs. (D10) and (D11). However, for $a > d/n$, these equations are effectively decoupled even for $a < d$ because the value $r^{-\tau^-_m} = 1 - a/d$ becomes exponentially small. Physically this corresponds to the situation when the size of the flux jet becomes larger than the film thickness, as discussed in Sec. II. The case of $a < d/n$ corresponds to a crossover from the restricted film geometry to a planar defect in an infinite superconductor considered in Ref. 23. Thus, to a very good approximation, we can set $r^{-\tau^-_m} = 0$ in Eqs. (D8) and (D10) if $a > d/n$. In this case Eq. (D11) gives an explicit expression for the parameter $r = J_0/J_m$ in terms of the coefficients $D^{(3)}_{2m-1}$, which are independent of $r$. To account for a non-zero value of $r^{-\tau^-_m}$, we solved Eqs. (D10) and (D11) iteratively. This self-consistency procedure is quite robust, typically converging within one or two iterations.

2. Convergence techniques

Streamline calculations are greatly accelerated by improving the convergence of hodograph series whose terms have the large-$m$ behavior $O(m^{-1})$. We find that the coefficients $C^{(1)}_{2m}$ and $D^{(2)}_{2m}$ are of this type, while the coefficients $C^{(2)}_{2m}$ and $D^{(3)}_{2m-1}$ converge more rapidly, as $O(m^{-3/2})$. Since the
asymptotic behaviors of $C_{2m}^{(1)}$ and $D_{2m}^{(2)}$ are computed directly from Eq. (D8), we can apply the method 1 of Appendix A.

We have also applied the convergence techniques to the matrix inversion procedure (D9) and (D10). As discussed in Appendix A, we exploit the correspondence between the $n$-series expansions for large $m$.

$$D_{2m-1}^{(3)} \rightarrow \gamma_D d_{2m-1}^{(3)}, \quad K_{m,m'} \rightarrow \frac{\sqrt{n} k_{m,m'}}{\sqrt{m}}, \quad (m \geq 1),$$

(D12)

where the lower- (upper-) case letters denote $n=1$ ($n>1$) quantities. The parameter $\gamma_D$ depends on $n$ (but not $m$).

We calculate $d_{2m-1}^{(3)}$ for $n=1$ exactly, using the method of analytic functions. For simplicity, we solve the aperture geometry, which is a limiting case of the constriction. However the aperture coefficients $d_{2m-1}^{(3)}$ have the same large-$m$ behavior as the constriction coefficients $C_{2m}^{(2)}$ and $D_{2m}^{(3)}$. In Fig. 5(a), the aperture is obtained by fixing the size of the opening $h$, but moving the channel wall to infinity ($a \rightarrow \infty$). This causes the lower hodograph region to vanish, corresponding to $r=0$.

The solution of the Maxwell equation $\nabla^2 \phi = 0$ for the complex potential $\phi = \phi - i \rho \psi$, satisfying all boundary conditions for the aperture, is given by

$$\phi = i \rho \left[ 1 - \frac{2}{\pi} \arccos \frac{z}{d-a} \right].$$

(D13)

$2I$ is the total current through the aperture of width $2(d-a)$. The transformation to hodograph coordinates is performed using the relation

$$J = J_x + i J_y = -\frac{1}{\rho} \frac{d \phi}{dz} = -\frac{2I}{\pi \sqrt{(d-a)^2 - z^2}},$$

(D14)

where we identify $J_m = 2I/\pi (d-a)$ as the current density at the center of the aperture ($z=0$). Using the identity $\psi = -\frac{\pi}{2} < \theta \leq \pi$

$$\psi = 1 - \frac{i}{\pi} \ln \left[ \frac{-\sqrt{1+(J_m/\bar{J})^2} - J_m/\bar{J}}{-\sqrt{1+(J_m/\bar{J})^2} + J_m/\bar{J}} \right].$$

(D15)

Equation (D15) is expanded as the following Taylor series for the case $J \geq J_m$:

$$\psi \frac{J}{I} = -1 + \sum_{m=1}^{\infty} \frac{2(-1)^m (r^{4m^2-1})}{\pi m} \left( \frac{J}{J_m} \right)^{4m^2-1} \sin(2m \theta).$$

(D16)

$$d_1^{(3)} = \frac{2}{\pi},$$

(D17)

for $m \geq 1$, we obtain the asymptotic behavior of $d_{2m+1}^{(3)} \approx 1/m^{3/2}$.

The matrix inversion procedure is now performed, replacing $d_m$ and $D_m$ by $d_{2m+1}$ and $D_{2m+1}$, and setting $\gamma_K = \sqrt{n}$ in Eqs. (A5)–(A7). The matrices $k_{m,m'}$ and $g_m$ are obtained from Eq. (D10) with $n=1$ and $r=0$.

$$k_{m,m'} = \frac{(-1)^{m+m'}}{2m' - 2m}, \quad g_m = \frac{(-1)^{m+1}}{2m}.$$

(D18)

The expression for $\gamma_D$ is given by $\gamma_D = Y_N(d_{2N-1}^{(3)} - X_N)$.

The self-consistency procedure starts with an initial estimate for $r$ (usually $r=0$). Then we calculate $D_{2m-1}^{(3)}$ using Eqs. (A5)–(A7). Finally we calculate $r$ from Eq. (D11), using convergence techniques of Appendix A. The procedure is repeated until convergence is achieved, usually within one or two iterations.

3. The limit $n \to \infty$

For finite $n$, the constriction solution requires a relatively sophisticated analysis as compared to the elbow and bridge geometries. However the case $n \to \infty$ stands out, since it permits analytical solutions for the constriction case as well. The $n \to \infty$ solutions can be obtained from the finite-$n$ solutions (D1)–(D8) by setting all $D_m$ coefficients to zero and using the limiting forms of Eq. (B8):
As for the bridge and elbow geometries, the singularity in $J(x,y)$ at the end of the crack disappears, along with the upper hodograph region. The simplified hodograph representation, shown in Fig. 6(b), no longer contains the Neumann BC, thus pving the way for a simple solution. On the horizontal line spanning the aperture ($y = 0$), the current density becomes uniform, with $J = J_0 \hat{y}$. Current conservation then specifies that $\gamma = J_0 / J_m = 1 - a_0 / d$, thus removing the need for a self-consistency procedure. Away from the crack, deviations from uniform current flow do not extend to infinity inside the channel. Instead, there exists a curve above which current flow is identically uniform, with $J = J_0 \hat{y}$. Similar to the asymmetric elbow, the region of current disturbance (i.e., nonuniform flow) takes the shape of a rectangle $W_n \times d$ as shown in Fig. 15.

4. Tail region

We now derive Eq. (52). In the tail region of Eq. (49) $J = J_0$, the first term in the summation contains the main dependence of $y$ on $J$. The other two terms in the equation can be replaced, using their values calculated when $J = J_0$. We expand the dominant term using Eq. (D8).

\[
\sum_{m=1}^{\infty} \left( -1 \right)^{m} \frac{2m D^{(2)} m}{2(m-1)} \left( \frac{J}{J_0} \right) \tau_{2m-1}^{-1} = \sum_{m=1}^{\infty} \frac{4}{\pi} \left( \frac{\tau_{2m}}{\tau_{2m} - \tau_{2m}^2} \left( \tau_{2m} - 1 \right) \right) \left( \frac{J}{J_0} \right) \tau_{2m-1}^{-1}
\]

\[
= \sum_{m=1}^{\infty} \frac{4}{\pi} \left( \frac{\tau_{2m}^2}{\tau_{2m}^2 - \tau_{2m}} \left( \tau_{2m} - 1 \right) \right) \left( \frac{J}{J_0} \right) \tau_{2m-1}^{-1} - \frac{1}{\pi m \sqrt{n}} \left( \frac{J}{J_0} \right)^{2m-1} \ln \left[ 1 - \left( \frac{J}{J_0} \right)^{2m-1} \right].
\]

(D21)

In the second equality, we have added and subtracted the large-$m$ behavior of the summand to improve the convergence properties using Eq. (B7).

Now we note that for $J = J_0$, successive terms in the summation have converged to their asymptotic large-$m$ behavior before the factors $J/J_0$ vary appreciably from 1. The logarithm term in Eq. (D21) therefore contains the main $J$ dependence. The other terms can be replaced, using their values calculated when $J = J_0$. To leading approximation then, we have

\[
y = W_n - \frac{d}{\pi \sqrt{n}} \ln \left[ 1 - \left( \frac{J}{J_0} \right)^{2m-1} \right].
\]

(D22)

Using $J = J_0$, we obtain the desired result, Eq. (52). When $n \to \infty$, only the first term in the summation of Eq. (D23) survives. By taking the appropriate limits, we obtain Eq. (53).
