Flux diffusion and the porous medium equation

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Abstract

Flux-flow problems in slabs and cylinders, and flux-creep problems in slabs, simplified, reduce to the porous medium equation with possible sign changes. The equation’s known self-similar solutions apply exactly or asymptotically if the boundary conditions are right. Flux-creep in cylinders corresponds to a modified porous medium equation that has different explicit solutions and different focusing solutions. © 1997 Elsevier Science B.V.

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1. Introduction

The porous medium equation is a non-linear diffusion equation that has been studied in relation to various physical problems or as a mathematical abstraction [1–5]. Many of its solutions are well known and categorised. Independently, two different problems of flux diffusion in superconductors have been described that both, after simplifications, reduce to the same equation. Where specific solutions were given, they were given in isolation without mentioning the porous medium analogy. The present article draws attention to some of the known porous medium solutions that can be applied to flux diffusion problems, and to other flux diffusion problems that do not reduce to any case previously investigated.

The porous medium equation can be written

\[
\frac{\partial G}{\partial t} = \Delta(G^{\sigma} G),
\]

where \( G(x,y,z,t) \) is a scalar variable and \( \sigma \) is a constant, here taken to be positive. In its simplest form, \( G \) is non-negative and the equation arises in several physical problems including gas diffusion in a porous medium. Then \( G \) represents the gas density. The equation can also be written

\[
\frac{\partial F}{\partial t} = \sigma \Delta F + (\nabla F)^2,
\]
where \( F = (1 + 1/\sigma)G \). \( F \) represents the pressure. For general \( \sigma \) values, Eq. (2) is simpler than Eq. (1) and its explicit solutions are usually simpler. For the diffusion of a perfect gas, \( \sigma = 1 \) and \( F = 2G \).

The geometries of foremost interest are the radial symmetries in \( d = 1 \), \( 2 \) or \( 3 \) dimensions (respectively planar, cylindrical and spherical symmetry). Then

\[
\frac{\partial G}{\partial t} = r^{d-2} \frac{\partial}{\partial r} \left[ r^{d-1} \frac{\partial}{\partial r} (|G|^\sigma G) \right],
\]

or

\[
\frac{\partial F}{\partial t} = (d - 1)\sigma + \frac{F}{r} \frac{\partial F}{\partial r} + \sigma F \frac{\partial^2 F}{\partial r^2} + \left( \frac{\partial F}{\partial r} \right)^2,
\]

where \( r = x, (x^2 + y^2)^{1/2} \) or \( (x^2 + y^2 + z^2)^{1/2} \).

Other problems include the spread of a viscous liquid on a horizontal surface, surface tension neglected. \( G(x,y,t) \) represents the height, \( \sigma = 3 \) and \( d = 1 \) or \( 2 \). The equation can describe radiative heat transfer and has notably been applied to the initial stage of an atmospheric nuclear explosion, with \( d = 3 \) and \( \sigma = 5 \). It is also encountered in plasma physics.

Bryksin and Dorogovtsev [6,7] studied the problem of a long superconductor in a flux-flow regime, in a magnetic field parallel to its length (and to a \( z \)-axis, suppose). The flux-flow resistivity, \( R \), for currents normal to \( O_2 \) is approximately \( \rho_n B_1/\mu_0 H_{c2} \) where \( B_1(x,y,z) \) is the flux density and \( \rho_n/\mu_0 H_{c2} \) is a constant of the material at given temperature. Then flux motion takes the form of Eq. (1) with \( \sigma = 1 \) and \( G = \rho_n B_1/2 \mu_0 H_{c2} \), or \( F = R/\mu_0 \). Principal geometries are the slab (\( d = 1 \)) and cylinder (\( d = 2 \)). Other shapes, particularly those with a high aspect ratio transverse to the applied field, require Eq. (1) within the sample to be solved simultaneously with non-trivial field equations (\( \text{curl} \ H = 0, \text{div} \ B = 0 \)) in the surrounding space. This has no porous medium analogue and the known solutions to Eq. (1) do not generally fit the complicated surface boundary conditions.

Vinokur et al. [8] considered the totally different problem of a long and wide superconducting slab in a flux-creep regime, in an applied magnetic field parallel to its length (and to \( O_2 \)). In effect the relation between electric field and current density was taken to be \( E_y = RJ_y \), where \( R \propto |J_y|^{\sigma} \) and, usually \( \sigma \gg 1 \). In reality, \( R \) depends on \( B \), as well as \( J_y \), but for small relative variations of \( B \), the adopted law is a good approximation. It leads to the porous medium equation again, but this time \( G(x,t) \) represents \( J_y(x,t) \) and \( d = 1 \). \( F \) again represents the resistivity:

\[
F = \frac{R}{q\mu_0} = \frac{1 + \sigma}{\sigma} \frac{E_y}{\mu_0 J_y} = \frac{1}{\sigma\mu_0} \frac{dE_y}{dJ_y}.
\]

For \( \sigma \gg 1 \) the profile of \( F \) is similar to that of \( E_y \) apart from possible sign changes, but is not like \( J_y \). This is connected with the finding emphasised by Gurevich [9,10] that \( E \) rather than \( B \) or \( J \) tends to have a universal, model-independent profile in this type of problem. The parameter \( q = \sigma/(1 + \sigma) \) introduced here, merely makes neater formulas.

Physical solutions to the porous medium equation share the following general features. There is always (for \( \sigma > 0 \)) an abrupt boundary (‘front’) between regions where \( F \) (and \( G \)) \( \neq 0 \) and where \( F \) (and \( G \)) \( = 0 \). \( F \) approaches zero at the front with finite slope \( \partial F/\partial r \) and the front advances at a velocity numerically equal to this slope. Solutions \( F(t,r) \) often evolve asymptotically towards self-similar forms, \( F = \delta^2 t^{-1/2} f(\rho) \). Here \( t \) is the time, possibly measured from an initialising event, \( \delta \) is a scaling length that varies as \( t^{\alpha} \) and \( \rho = r/\delta \). Such self-similar solutions were studied by Barenblatt [11,12] and others [13–15]. In some cases \( \alpha \) can be deduced immediately from the boundary conditions and is a simple explicit function of \( \sigma \) (problems of the first kind in Barenblatt’s classification) and in others it has to be calculated (second kind). Sometimes \( f(\rho) \) is also a simple explicit function.
Table 1
Explicit solutions to Eq. (6)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$f(r)$</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1 - \rho$</td>
<td>‘Linear pressure’ solution</td>
</tr>
<tr>
<td>$1 + \sigma$</td>
<td>$\alpha \rho^3$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2 + \sigma}$</td>
<td>Constant</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$1 - \rho^2$</td>
<td>Barenblatt or Barenblatt–Pattle solution</td>
</tr>
<tr>
<td>$\frac{1}{2(1 + \sigma)}$</td>
<td>$\frac{\rho^3 - \rho^2}{2(2 + \sigma)}$</td>
<td>‘Dipole’ solution</td>
</tr>
<tr>
<td>Any</td>
<td>$\frac{-\rho^2}{2(2 + \sigma)}$</td>
<td>‘Quadratic pressure’ solution</td>
</tr>
</tbody>
</table>

In three cases where $f(\rho)$ changes sign at some nonzero $\rho$ value, an integration constant has been chosen so this happens at $\rho = 1$. The quadratic pressure solution is better expressed $F = (4 + 2\sigma)^{-1}(1 - \sigma)^{-1}r^2$; parameters $\delta$ and $\alpha$ are redundant.

The remainder of this paper is organized as follows. In Section 2, self-similar solutions for planar geometry are summarily reviewed, followed by their application to the porous medium and flux diffusion problems. Periodic solutions are reserved for Section 3. Section 4 deals with the porous medium and flux-flow problems in cylindrical geometry, but the flux creep problem in cylindrical geometry is a case apart and is left till Section 5.

2. Planar self-similar solutions

With $d = 1$ and the substitution $F = \delta^2 t^{-1}f(\rho)$, Eq. (4) becomes

$$\sigma f \frac{d^2 f}{d \rho^2} + \left( \frac{df}{d \rho} \right) + \alpha \rho \frac{df}{d \rho} + (1 - 2\alpha) f = 0.$$ (6)

For the problems envisaged it is sufficient to consider $\rho \geq 0$ but while $F$ is necessarily positive, $f$ may take either sign, like $t$ depending on whether the origin of time is to be taken in the past or the future. Eq. (6) is quadratic and normally has two solutions for given $\sigma$ and $\alpha$, each of which can be modified by the choice of integration constants. For certain $\alpha$ values, one of the solutions can be expressed as a simple explicit function. Such cases are listed in Table 1. Where $f(\rho)$ changes sign at $\rho = 1$, this is an oversimplification because $df/d\rho$ need not be continuous at that point. Physical problems invariably correspond to $f = f(\rho)$ as tabulated for $0 < \rho < 1$ and $f = 0$ for $\rho > 1$ or vice versa.

2.1. Application to selected porous slab problems

A practical arrangement for $d = 1$ porous medium studies would consist of a long tube open at one end, subject to external pressures. Gas could also be injected at the closed end. But the present purpose is to pursue the analogy with a superconducting slab, so a slab of porous medium will be imagined. The same pressure is applied each side, and if any gas is injected, it is injected into the median plane, so in all cases $G$ and $F$ are symmetric about that plane, and only half the slab need be examined. A list of selected idealised problems follows. Unless stated to the contrary the initial condition is no gas and no external pressure.
**PS1** Starting at a given moment, \( t_1 \), the external pressure is ramped steadily. Gas has not yet reached the median plane but it will do so at a later time, \( t_Z \).

**PS2** As PS1, but after \( t_Z \) when the fronts have met at the median plane.

**PS3** A fixed external pressure is applied suddenly at \( t_1 \). Gas has not yet reached the median plane.

**PS4** As PS3, but the external pressure is only applied momentarily, at \( t_k \) so that later, while the fronts are progressing inwards, gas is also escaping back out to vacuum.

**PS5** From a given moment, \( t_1 \), gas is pumped steadily into the median plane. Gas has not yet reached the surfaces.

**PS6** As PS5 but later, when gas is escaping to vacuum.

**PS7** At \( t_1 \) a fixed quantity is released suddenly into the median plane. It has not yet reached the surfaces.

**PS8** As PS7 but later when gas is escaping to vacuum.

**PS9** Initially the slab was uniformly permeated with gas in equilibrium with an external pressure then at \( t_0 \) the external pressure is suddenly reduced to zero.

The relevant self-similar solutions are given in Table 2, and they require the following comments. Typically, the two dissimilar solutions for a same \( \alpha \) value apply at different stages of a same process (PS1 then PS2, PS5 then PS6 and also PS3 then its sequel when the solution \( f = \text{constant} \) is rapidly approached). In problem PS1 one could have \( \rho = 0 \) at the surface, and then the origin of time would have to be at the start of the ramp, at \( t_1 \). One would have \( f(\rho) = 1 - \rho \) in the interval \( 0 < \rho < 1 \) that corresponds to the penetrated region (it grows linearly with \( t \)) and \( f(\rho) = 0 \) for \( \rho > 1 \). Alternatively, one could choose \( \rho = 0 \) at the median plane and \( t = 0 \) at the (future) instant, \( t_Z \) when the fronts meet there. Then \( f(\rho) = 0 \) for \( 0 < \rho < 1 \) (this interval decreases linearly with \(-t\)) and \( f(\rho) = 1 - \rho \) for \( \rho > 1 \) out to the surface. The latter choice coincides with the origins that must necessarily be used for problem PS2. In fact, for PS2 one could also define \( \delta \) so that \( |d\delta/dt| \) has the same value.

### Table 2

<table>
<thead>
<tr>
<th>Problem</th>
<th>Type</th>
<th>( \alpha )</th>
<th>( f )</th>
<th>Origin of ( \rho )</th>
<th>Origin of ( t )</th>
<th>X/A</th>
</tr>
</thead>
<tbody>
<tr>
<td>PS1</td>
<td>I</td>
<td>1</td>
<td>((1 - \rho) \text{ or } 0)</td>
<td>(a)</td>
<td>(a)</td>
<td>X</td>
</tr>
<tr>
<td>PS2</td>
<td>I</td>
<td>1</td>
<td>( f_\delta(\rho) )</td>
<td>mp</td>
<td>( t_2 )</td>
<td>A</td>
</tr>
<tr>
<td>PS3</td>
<td>I</td>
<td>( \frac{1}{2} )</td>
<td>( f_\delta(\rho) )</td>
<td>( 0 &lt; \rho &lt; 1 )</td>
<td>s</td>
<td>( t_3 )</td>
</tr>
<tr>
<td>PS4</td>
<td>II</td>
<td>( \frac{1}{2 + 2\alpha} )</td>
<td>( \rho^\alpha - \rho^2 )</td>
<td>( 0 &lt; \rho &lt; 1 )</td>
<td>s</td>
<td>( t_4 )</td>
</tr>
<tr>
<td>PS5</td>
<td>I</td>
<td>( \frac{1 + \alpha}{2 + \alpha} )</td>
<td>( f_\delta(\rho) )</td>
<td>( 0 )</td>
<td>( \rho &gt; 1 )</td>
<td>mp</td>
</tr>
<tr>
<td>PS6</td>
<td>I</td>
<td>( \frac{1 + \alpha}{2 + \alpha} )</td>
<td>( \alpha \rho^\alpha )</td>
<td>( 0 )</td>
<td>( \rho &gt; 1 )</td>
<td>arbitrary</td>
</tr>
<tr>
<td>PS7</td>
<td>I</td>
<td>( \frac{1}{2 + \alpha} )</td>
<td>( \frac{1 - \rho^2}{4 + 2\alpha} )</td>
<td>( 0 )</td>
<td>( \rho &gt; 1 )</td>
<td>mp</td>
</tr>
<tr>
<td>PS8</td>
<td>I</td>
<td>0</td>
<td>( f_\delta(\rho) )</td>
<td>arbitrary</td>
<td>(b)</td>
<td>A</td>
</tr>
<tr>
<td>PS9</td>
<td>I</td>
<td>0</td>
<td>( f_\delta(\rho) )</td>
<td>arbitrary</td>
<td>(b)</td>
<td>A</td>
</tr>
</tbody>
</table>

\( f(\rho) \) represents the gas pressure profile across one half of the porous slab. X signifies that the self-similar solution is the exact solution to the idealised problem listed, A that it only an asymptotic solution, mp = median plane, s = surface. \( f_\delta(\rho) \), \( f_\delta(\rho) \), \( f_\delta(\rho) \) and \( f_\delta(\rho) \) do not have simple expressions as in Table 1.

(a) Arbitrary but linked; see text, (b) the origin of \( t \) has to be determined numerically for each \( \alpha \) value.
as in PS1. \( f_s(\rho) \) resembles a hyperbola, with
\[
f_s(\rho) = f_s(0) + \rho^2/2\sigma + \cdots \quad \text{(for } \rho \ll 1 \text{) and } f_s(\rho) \equiv \rho \quad \text{(for } \rho \gg 1 \text{).}
\]
However, while the self-similar solution to PS1 is the exact solution, the PS2 solution is only valid asymptotically for a short instant after \( t_1 \) until the external boundary conditions again become important.

\( f_s(\rho) \) and \( f_s(\rho) \) are monotonically decreasing parabola-like functions that are exact solutions to the idealised listed problems. However, if it takes a finite time to build up the pressure in problem PS3, the solution \( f_s(\rho) \) is only later approached asymptotically. \( f_s(\rho) \) and \( f_s(\rho) \) were drawn for the particular case \( \sigma = 1 \) (as curves (c) and (b) respectively in Fig. 2) in an earlier publication [16]. Problem PS4 is technically of the second kind since there is no imposed parameter (such as quantity of gas) that would give \( \sigma \) immediately by dimensional analysis. But the ‘dipole solution’ is the only front solution that has a zero at the origin and no other zero in the interval \( 0 < \rho < 1 \). It would be the exact solution to the idealised problem in which the pressure pulse was really instantaneous, but otherwise is approached asymptotically. The self-similar solution to problem PS6 describes steady gas flow [2], and is approached asymptotically after the gas has been escaping for some time.

The profile \( f_s(\rho) \) results as a long time asymptotic solution to problems PS8 and PS9 and after various other initial conditions, where gas is escaping to vacuum. \( F(r,t) \) can be factored into this spatial profile and a temporal decay function \( t^{-1} \), but the origin of \( t \) has to be found numerically for each problem and each \( \sigma \) value. As a rule the starting conditions do not correspond to the profile \( f_s(\rho) \). One might choose to put \( \rho = 0 \) at the surface and \( \rho = 1 \) at the median plane. There is an algebraic link between the two ends of the profile, viz:

\[
f_s(\rho) = f_s(1) - \frac{1}{2\sigma}(1-\rho)^2 + \cdots \quad (\rho \to 1)
\]

\[
= \left( \frac{(2 + 2\sigma)\rho}{\sqrt{4 + 2\sigma}f_s(1)} \right)^q f_s(1) + \cdots \quad (\rho \to 0),
\]

but \( f_s(1) \) is not expressible as an elementary function of \( \sigma \). For comparison it should be mentioned that if the final applied pressure is nonzero, the final uniform pressure profile is approached exponentially as in normal diffusion. The ‘quadratic pressure’ solution does not correspond to a plausible boundary condition.

2.2. Application to flux-flow slab problems

The problems PS1–9 and their solutions can all be reformulated to apply to a superconducting slab in which the flux-flow law holds. It is sufficient to put \( \sigma = 1 \) and substitute flux density for gas density, external applied field for external applied gas pressure and flux injection for gas injection. However, there are a series of other cases with no porous medium analogue – those where \( B_1 \) changes sign. Hulshof [4] made a systematic study of self-similar solutions to Eq. (3) with \( d = 1 \) and sign changes of \( G \), but flux-flow superconductors provide a first application of his results to a physical problem. To summarise, he found solutions with fronts at \( \rho = \pm 1 \) and any number, \( n \) of intermediate zeros. The Barenblatt solution is the one with \( n = 0 \), the ‘dipole solution’ the one with \( n = 1 \) and if \( f_s(\rho) \) is continued so as to be periodic, and is rescaled, it is the case \( n \to \infty \). A problem can also be imagined that has the \( n = 2 \) solution. Flux is released into the median plane, then a moment later an equal and opposite flux. This is a second-kind problem. As reported earlier [16], \( \alpha = 0.1891 \), and for general \( \sigma \) values it is always close to \( \frac{1}{4}(2 + \sigma)/(1 + \sigma)^2 \). At the intermediate zeros, \( f(\rho) \) always approaches the axis as \( |\rho - \rho_0|^{\alpha} \). Physically, there is flux transport or annihilation at such a point, but not at a flux-front. Such solutions are unstable, and if there is any imbalance between the positive and negative flux the sign reversals will vanish in finite time and the system evolve towards the Barenblatt solution [17].

2.3. Application to selected flux-creep slab problems

The problems PS1–9 can be reformulated substituting magnetic flux for gas, but in the flux-creep condition the \( G \) parameter represents the spatial derivative of flux density, not flux density itself. Accordingly, all the
boundary conditions on $G$ are changed, as the following list and Table 3 testify. Initially the slab always has a uniform flux density in equilibrium with a uniform applied field. It is excited symmetrically so that $B_z$ is symmetric, but $J_z$ antisymmetric about the median plane.

**CS1** Starting at $t_1$, the field is ramped steadily. The disturbance has not yet reached the median plane. The fronts will meet later, at $t_2$.

**CS2** The same, after $t_2$ when the fronts have met.

**CS3** At $t_1$ the applied field is suddenly changed to a new value. The disturbance has not yet reached the median plane.

**CS4** Starting at $t_1$, additional flux is pumped steadily into the median plane. The disturbance has not yet reached the surfaces.

**CS6** The same, later when excess flux is escaping.

**CS7** At $t_1$ a given amount of additional flux is released suddenly into the median plane. The disturbance has not yet reached the surfaces.

**CS8** The same, later when excess flux is escaping.

**CS9** The applied field was changed suddenly as in CS3, but the disturbance has now reached the median plane.

This time the ‘linear pressure solution’ corresponds to improbable physical boundary conditions, not listed. $f_1(\rho)$, $f_2(\rho)$ and $f_3(\rho)$ are the same functions as for Table 2, but wherever the origin of $\rho$ was at the median

Table 3
Self-similar solutions to listed flux-creep slab problems

<table>
<thead>
<tr>
<th>Problem</th>
<th>Type</th>
<th>$\alpha$</th>
<th>$f$</th>
<th>Origin of $\rho$</th>
<th>Origin of $t$</th>
<th>$X/A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CS1</td>
<td>I</td>
<td>$\frac{1+\alpha}{2+\alpha}$</td>
<td>$f_1(\rho)$</td>
<td>$0 &lt; \rho &lt; 1$</td>
<td>$s$</td>
<td>$t_1$</td>
</tr>
<tr>
<td>CS2</td>
<td>I</td>
<td>$\frac{1+\alpha}{2+\alpha}$</td>
<td>$\rho^s$</td>
<td>$\rho &gt; 1$</td>
<td>mp</td>
<td>arbitrary</td>
</tr>
<tr>
<td>CS3</td>
<td>II</td>
<td>$\frac{1}{2+\alpha}$</td>
<td>$\frac{1-\rho^2}{4+2\alpha}$</td>
<td>$0 &lt; \rho &lt; 1$</td>
<td>$s$</td>
<td>$t_3$</td>
</tr>
<tr>
<td>CS4</td>
<td>II</td>
<td>$\alpha(\sigma)$</td>
<td>$f_2(\rho)$</td>
<td>$0 &lt; \rho &lt; 1$</td>
<td>$s$</td>
<td>$t_4$</td>
</tr>
<tr>
<td>CS5</td>
<td>I</td>
<td>$\frac{1}{\beta}$</td>
<td>$f_2(\rho)$</td>
<td>$0 &lt; \rho &lt; 1$</td>
<td>mp</td>
<td>$t_5$</td>
</tr>
<tr>
<td>CS6</td>
<td>I</td>
<td>$\frac{1}{\beta}$</td>
<td>constant</td>
<td>$\rho &gt; 1$</td>
<td>arbitrary</td>
<td>arbitrary</td>
</tr>
<tr>
<td>CS7</td>
<td>I</td>
<td>$\frac{1}{2+2\alpha}$</td>
<td>$\frac{\rho^s-\rho^2}{4+2\alpha}$</td>
<td>$0 &lt; \rho &lt; 1$</td>
<td>mp</td>
<td>$t_7$</td>
</tr>
<tr>
<td>CS8</td>
<td>I</td>
<td>0</td>
<td>$f_3(\rho)$</td>
<td>arbitrary</td>
<td>(a)</td>
<td>A</td>
</tr>
<tr>
<td>CS9</td>
<td>I</td>
<td>0</td>
<td>$f_4(\rho)$</td>
<td>arbitrary</td>
<td>(a)</td>
<td>A</td>
</tr>
</tbody>
</table>

$f(\rho)$ represents the profile of $J_z$ across one half of the slab. $X$ signifies that the self-similar solution is the exact solution to the idealised problem listed, $A$ that it is only an asymptotic solution, $mp$ = median plane, $s$ = surface. $f_1(\rho)$, $f_2(\rho)$, $f_3(\rho)$ and $f_4(\rho)$ do not have simple expressions as in Table 1.

(a) The origin of $t$ has to be determined numerically for each $\sigma$ value.
plane it is now at the surface and vice versa. CS3 is the problem solved by Vinokur et al. [8], and the listed solution is their solution. \( f_3(p) \) is the Hulshof \( n = 2 \) solution. CS7 is a problem of Barenblatt’s first kind since the conservation of flux immediately gives the value of \( \alpha \), while the porous slab problem (PS4) with the same solution was of the second kind. Solution CS2 is the solution for \( \partial E_x / \partial x = \partial B_y / \partial t \) = constant, and corresponds to a steady current flow. Unlike the solution PS2 it does not apply immediately after \( t_s \) but is later approached asymptotically. The trivial solution CS6 corresponds to steady uniform current flow with \( \partial E_x / \partial x = \partial B_y / \partial t = 0 \). Any transient deviation from this decays exponentially in time.

3. Periodic planar solutions

For \( d = 1 \), the porous medium equation has a different type of self-similar solution that is periodic in time, having \( G(r, t + \pi / \omega) = -G(r, t) \). It is a front solution and has \( F(r, t) = \omega r^2 f(\phi) \), where \( \phi = \omega t + \lambda \ln(r / r_0) \) and \( \lambda \) is a numerical constant to be found. Just as the partial differential Eq. (4) reduced to the ordinary differential Eq. (6) with \( \alpha \) as a parameter (a nonlinear eigenvalue), so now it reduces to an ordinary differential equation for \( f(\phi) \) with \( \lambda \) as a parameter. This is always a Barenblatt problem of the second kind since the value of \( \lambda \) is never a priori obvious. In practice it is preferable to solve for the better-behaved function \( h(\phi) \), where \( d h / d \phi = (qf)^{1/2} \text{sgn}(G) \). Then:

\[
\lambda^2 \frac{d^2 h}{d \phi^2} + \left( \frac{4\lambda}{q} - \lambda - \frac{d h}{d \phi} \right)^{q-2} \frac{d h}{d \phi} + \left( \frac{4}{q^2} - \frac{2}{q} \right) h = 0.
\]

(7)

The method is to put \( \lambda = 1 \) and find the period. The period scales as \( \lambda \), so it can be made \( 2\pi \), and \( h \) scales as \( \lambda^{-1/\sigma} \). We recently applied this result to flux-creep [18]. Then \( h \) represents the magnetic flux between the reference plane and the front (extinction) plane and \( d h / d \phi \) the electric field at the reference plane. With increasing \( \sigma \) the self-similar waveforms become more markedly non-sinusoidal. Rather than \( \lambda \) as a function of \( \sigma \) it is more instructive to plot \( \sigma \lambda / (2 + \sigma) \) since this is the ratio of phase lag, \( d \phi / d r \) to attenuation of the alternating flux density. As Fig. 1 shows, the ratio only differs markedly from 1 (the value for normal or Ohmic diffusion) at very high \( \sigma \) values, where it falls. At first sight this is counterintuitive since \( \sigma \to \infty \) corresponds to

![Figure 1](https://via.placeholder.com/150)

Fig. 1. Parameter occurring in the self-similar periodic solution to the porous medium equation. If a periodically changing flux-density is penetrating into a superconducting slab obeying a flux-creep law, the ratio of phase-lag to attenuation before complete extinction is given by the plotted parameter \( \sigma \lambda / (2 + \sigma) \). \( q \) or \( \sigma \) indicates the degree of nonlinearity. An Ohmic conductor would have \( \sigma = q = 0 \), a Bean superconductor \( \sigma = \infty \) and \( q = 1 \). The dotted curve shows for comparison the function \((1 - q / 2)^{-1/2}\) that corresponds to a small rotating flux-density in the presence of a larger steady perpendicular flux-density, \( B_y \). This has an explicit solution \( h \propto e^{b} \) as discussed earlier [18].
the Bean model, and the traditional picture [19] shows a sign-reversal of the current density progressing inwards over half a cycle. But the self-similar waveform for \( h \) resembles a square wave, and much of the change occurs in a short part of each half-period.

We showed numerically [18] that if the applied periodic field at the surface is sinusoidal, as it progresses and is attenuated, the periodic flux-density gradually becomes less sinusoidal and resembles the self-similar form. However, since the measured response (complex susceptibility and harmonics) depends primarily on the transition region where the amplitudes are highest, a numerical calculation like Rhyner’s [20] or ours [18] is still necessary, whether or not the disturbance penetrates the entire slab.

4. Cylindrical self-similar solutions

Rewriting Eq. (6) for \( d = 2 \) involves adding a term \( \sigma f / \rho \cdot df / d\rho \). Three of its explicit solutions have their equivalent in two or more dimensions. There is \( \alpha = 1/2 \) with \( f(\rho) = \text{constant} \), \( \alpha = 1/(2 + 2\sigma) \) with \( f = (1 - \rho^2)/(4 + 4\sigma) \) and \( f = -\rho^2/(4 + 4\sigma) \) for which \( \alpha \) is redundant.

4.1. Application to porous cylinder or cylindrical flux-flow superconductor

There is one distinctive new problem that does not arise in planar geometry. Before discussing it the others will be summarily dismissed. For a start there are the source problems in which gas, or flux, is injected along the axis and is spreading out but has not yet reached the surface. As before, in problem PS7, the Barenblatt solution applies, in its \( d = 2 \) version if a fixed amount of gas is suddenly injected. If gas were to be continually pumped into the axis, the pressure profile would have a logarithmic singularity at \( \rho = 0 \). In fact the cylinder would need to have a specified inner radius. No matter how the gas was initially introduced, if in the long term it is escaping out to vacuum, a solution with \( \alpha = 0 \) is approached asymptotically, similar to the one for problems PS8 and 9.

That leaves the in-flow problems where an initially gas-free cylinder is exposed to an external pressure. For a first moment of shallow penetration, the cylinder radius is virtually infinite, and the analogous \( d = 1 \) solution applies. There follows a longer stage when there can be no self-similar solution, but later, when the unpenetrated region has very much shrunk and is about to disappear altogether, the details of its shrinkage and

Fig. 2. Critical exponent in the self-similar solution to the focusing problem in cylindrical geometry. The full round points are from Table 1 in Ref. [5], but note that the present \( \alpha \) is the reciprocal of \( \alpha \) in [5]. Results are also shown for the analogous flux-creep problem.
disappearance can be analysed by again taking the cylinder radius to be virtually infinite. This is the focusing problem treated by Aronson and Graveleau [5], who found a self-similar solution with the following characteristics. δ is equated to the front radius, so that f(ρ) = 0 for ρ < 1 and f(ρ) < 0 for ρ > 1. Since the external boundary is far away, F(r,t) cannot change appreciably on the short time scale here relevant. This means δ²⁻¹ f(ρ) must be time-invariant at high ρ values, so f(ρ) must vary as (−t)²⁻κ, and so also as ρ²⁻κ. The problem is of Barenblatt's second kind. Aronson and Graveleau tabulated α for different d and σ values. For d = 1 and all σ values α = 1 (cf. problem PS1) but for d = 2 or more, α < 1 as shown in Fig. 2. It means the front accelerates and its velocity weakly diverges. There seems to be little doubt that this self-similar solution would be approached asymptotically at the relevant place and time [21].

After the disappearance of the flux-front, the problem analogous to PS2 arises but this remains a problem of the first kind, since α is determined by the power-law profile f(ρ) inherited from the preceding focusing problem.

5. Cylindrical flux-creep problems

In these problems a long cylinder (axis Oz) has an axial flux density, Bz(r,t) and a circulating current density, Jr(r,t). Jr(r,t) can be represented by a scalar, G(r,t) but when applying the Laplacian it is not permissible to forget it is a circulating vector component, and instead of Eqs. (3) and (4) there results for d = 2:

\[ \frac{\partial G}{\partial \tau} = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r|G|^n G) \right) \]  

and

\[ \frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial r} \frac{\partial F}{\partial r} + \sigma F \frac{\partial^2 F}{\partial r^2} + \left( \frac{\partial F}{\partial r} \right)^2 - \frac{\sigma^2 + F^2}{1 + \sigma r^2} \]  

Eq. (6) becomes

\[ \alpha \frac{d^2 f}{d \rho^2} + \sigma f \frac{df}{d \rho} + \left( \frac{df}{d \rho} \right)^2 - q \sigma \frac{f^2}{\rho^2} + \alpha \rho \frac{df}{d \rho} + (1 - 2 \alpha) f = 0. \]  

Its explicit solutions are listed in Table 4. Respectively they correspond to the steady inward transport of flux, analogue to CS2, the steady outward transport of flux, analogue to CS6, a boundary condition in which flux is conserved, B(0) fixed. Spreading flux, \( B_z(0) \) fixed. Spreading flux (total flux conserved).

<table>
<thead>
<tr>
<th>α</th>
<th>f(ρ)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + α</td>
<td>( \alpha \rho^\gamma )</td>
<td>Steady inward flux motion</td>
</tr>
<tr>
<td>2 + α</td>
<td>( \alpha \rho^{-\gamma} )</td>
<td>Steady outward flux motion</td>
</tr>
<tr>
<td>1 + σ</td>
<td>( \alpha \rho^{-\gamma} )</td>
<td></td>
</tr>
<tr>
<td>1 + 3α</td>
<td>( \frac{1 + \sigma}{(2 + 3\sigma)(2 + \sigma)} (\rho^{-\gamma} - \rho^2) )</td>
<td>Spreading flux, ( B_z(0) ) fixed</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1 + \sigma}{(2 + 3\sigma)(2 + \sigma)} (\rho^2 - \rho^2) )</td>
<td>Spreading flux (total flux conserved)</td>
</tr>
<tr>
<td>Any</td>
<td>( \frac{1 + \sigma}{(2 + 3\sigma)(2 + 3\sigma)} \rho^2 )</td>
<td>No plausible condition</td>
</tr>
</tbody>
</table>

In two cases an integration constant has been chosen so f(ρ) changes sign at ρ = 1. The physical problems listed correspond to f = f(ρ) as tabulated for 0 < ρ < 1 and f ≠ 0 for ρ > 1.
pumped into the cylinder axis at such a rate as to maintain a constant flux-density there (the disturbance has not yet reached the outer radius) and a fixed amount of flux released suddenly on the axis (disturbance not yet reached outer radius), analogue to CS7.

The other case most worth investigating is the focusing problem, to check whether the flux-front accelerates before vanishing. It cannot be assumed without proof that Eq. (9) has self-similar solutions analogous to the Graveleau [5] solutions to Eq. (4) but a numerical investigation of Eq. (10) suggests it does. The critical $\alpha$ value (Fig. 2) never falls below 0.9 so the flux-front velocity diverges very weakly, especially if $\sigma \gg 1$ and its divergence could scarcely be observed experimentally.

6. Conclusions

Previous work on the porous medium equation applies most directly to long superconducting shapes in a longitudinal applied field, where the demagnetising effect can be neglected. Various self-similar solutions then apply, exactly or asymptotically. In all other cases, and especially for transverse films, discs and rings the surface boundary conditions are governed by electromagnetic field equations that have no analogue in gas or heat diffusion so the porous medium solutions for the same geometries would not apply. Even with a long cylinder, allowance must be made that the resistivity obeys Eq. (9) and not Eq. (4). This means that the flux-front velocity would diverge too weakly, when approaching the cylinder axis, for this to be observable.

7. Added note

Bass, Shapiro and Shvartser [22] have also applied the Barenblatt–Pattle solutions for $\sigma = 1$ and $d = 1$ and 2 to flux flow.

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References