Self-field hysteresis loss in periodically arranged superconducting strips

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Received 16 May 1997; accepted 3 July 1997

Abstract

The magnetic field and current distributions inside superconducting strips arranged in a $z$-stack and an $x$-array are calculated analytically using the transformation method proposed by Mawatari for the case where a transport current is passed through the strips. The magnetic field distributions are used to calculate the self-field ac hysteresis losses of both the $z$-stack and the $x$-array. The self-field ac hysteresis loss of the $z$-stack is always larger than the loss of the $x$-array. © 1997 Elsevier Science B.V.

PACS: 85.25Kk; 84.70.+p; 74.76.Bz

1. Introduction

The critical state model [1,2] has been used extensively to describe the electrodynamics of type II superconductors and to calculate the ac hysteresis loss. Calculations were first performed for the simple cases of long superconducting cylinders and infinite superconducting slabs in parallel magnetic fields. The more complicated case of the critical state in a flat superconducting strip was first investigated by Swan [3] and Norris [4]. High-temperature superconductors (HTS) have renewed the interest in the critical state of flat superconductors with demagnetizing factors close to 1 [5,6]. Examples are HTS single crystals which are thin along the $c$-direction and wide in the $ab$-directions used in experiments on flux vortex dynamics as well as Bi-2223/Ag tapes [7] and YBCO/Ni (hastelloy) thin-film flexible tapes [8] to be used in power applications. In most applications, the tapes will be used in array or stack configurations were the strips couple electromagnetically. The ac hysteresis loss of multiple strips is expected to be quite different from that of an isolated strip, and it is therefore important to investigate the behaviour of superconducting strips in array and stack configurations.

Mawatari [9,10] recently calculated the magnetic field distribution $H(x)$ and the current density distribution $J(x)$ ($x$ is the coordinate along the width of the strip) for a $z$-stack and an $x$-array of strips where an ac magnetic field is applied perpendicular to the strips. Mawatari [9] showed that, for both the $x$-array and the $z$-stack, the $H$--$J$ relation of Biot–Savart’s law can be transformed into the $H$--$J$ relation of an...
isolated strip, \( H(x) \) and \( J(x) \) for multiple strips can
then be obtained from the \( H(x) \) and \( J(x) \) of the
isolated strip by a simple transformation.

In the following paper we adopt the transformation
method developed by Mawatari \(^9\) to calculate
the magnetic field distribution \( H(x) \) and the current
density distribution \( J(x) \) for an \( x \)-array and a \( z \)-stack
for the important cases where dc or ac transport
currents are flowing through the strips. From \( H(x) \)
we calculate the self-field ac hysteresis losses of the
\( z \)-stack and the \( x \)-array.

2. The self-field critical state in a periodic array
of superconducting strips

2.1. The \( z \)-stack

The magnetic field \( H(r) \) at position \( r \) created by
a current distribution \( J(r') \) is given by Biot–Savart’s
law

\[
H(r) = \frac{1}{4\pi} \int J(r') \times \frac{r - r'}{|r - r'|^3} \, d^3r'.
\]  

(1)

Let us consider a single infinitely long superconduct-
ing strip of width 2\( a \) along the \( x \)-direction, thickness \( d \) along the \( z \)-direction (\( a \gg d \)), positioned at the coordinate origin. In this case one obtains from Eq. (1)

\[
H_z(x,z) = -\frac{d}{2\pi} \int_{-a}^{a} \tilde{J}(u) \frac{x-u}{(x-u)^2 + z^2} \, du.
\]  

(2)

where \( \tilde{J}(u) \) is the current density in the supercon-
ducting strip averaged over the thickness \( d \) of the
strip,

\[
\tilde{J}(x) = \frac{1}{d} \int_{-d/2}^{d/2} J_z(x,z') \, dz'.
\]  

(3)

Let us now consider an infinite number of
infinitely long strips periodically stacked above each
other along the \( z \)-direction with a spacing \( D \) where
\( D > d \). Such a \( z \)-stack of strips is shown in Fig. 1.

The \( z \)-component of the magnetic field, \( H_z(x) \) across
the strip, is given by

\[
H_z(x) = -\frac{d}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-a}^{a} \tilde{J}(u) \frac{x-u}{(x-u)^2 + (nD)^2} \, du.
\]  

(4)

or \(^{9,11}\)

\[
H_z(x) = -\frac{d}{2D} \int_{-a}^{a} \tilde{J}(u) \coth \left( \frac{\pi(x-u)}{D} \right) \, du.
\]  

(5)

Introducing the variable transformations

\[
\tilde{x} = \frac{D}{\pi} \tanh \left( \frac{\pi x}{D} \right), \quad \tilde{u} = \frac{D}{\pi} \tanh \left( \frac{\pi u}{D} \right)
\]

and \( \tilde{a} = \frac{D}{\pi} \tanh \left( \frac{\pi a}{D} \right) \)

(6)

one obtains for Eq. (5) using \( \tilde{J}(u) = \tilde{J}(-u) \)

\[
H_z(x) = \tilde{H}_z(\tilde{x}) = -\frac{d}{2\pi} \int_{-\tilde{a}}^{\tilde{a}} \tilde{J}(\tilde{u}) \, d\tilde{u},
\]  

(7)

where \( \tilde{J}(\tilde{u}) = \tilde{J}(u) \).

Eq. (7) has the same form as Eq. (2) for \( z = 0 \) which describes \( H_z(x) \) of a single strip. The current
and magnetic field distributions for a single strip have been discussed by Norris [4], Brandt et al. [5] and Zeldov et al. [6]. The major problem solved by these authors was to find a current distribution which produces a zero $z$-component of the magnetic field inside the superconducting strip for $|x| < b$ and a critical state magnetic field distribution for $b < |x| < a$, where $b$ represents the distance of the flux front from the centre of the strip. From their calculations it follows that for the $z$-stack

$$
\tilde{H}_{z}(\bar{x}) = \begin{cases} 
0, & |\bar{x}| < \tilde{b}, \\
-H_{z}[\bar{x}] \tanh \left( \frac{\bar{x}^2 - \tilde{b}^2}{\tilde{a}^2 - \tilde{b}^2} \right)^{1/2}, & \tilde{b} < |\bar{x}| < \tilde{a}, \\
-H_{z}[\bar{x}] \tanh \left( \frac{\bar{x}^2 - \tilde{b}^2}{\tilde{a}^2 - \tilde{b}^2} \right)^{1/2}, & |\bar{x}| > \tilde{a},
\end{cases}
$$

(8)

where

$$\tilde{b} = \frac{D}{\pi} \tanh \left( \frac{\pi b}{D} \right) \quad \text{and} \quad H_{z} = J_{c} d/\pi.
$$

(9)

Here, $J_{c}$ is a field-independent critical current density. (The more complicated case of a field-dependent $J_{c}$ has been discussed in Ref. [12].)

For $\tilde{J}(\bar{x})$ one obtains from [5]

$$\tilde{J}(\bar{x}) = \begin{cases} 
\frac{2 J_{c}}{\pi} \tanh \left( \frac{\tilde{b}^2 - \bar{x}^2}{\tilde{a}^2 - \tilde{b}^2} \right)^{1/2}, & |\bar{x}| < \tilde{b}, \\
J_{c}, & \tilde{b} < |\bar{x}| < \tilde{a}, \\
0, & |\bar{x}| > \tilde{a}.
\end{cases}
$$

(10)

Let us assume that after zero-field cooling a current $I_{dc}$ is applied to each strip where $I_{dc} < I_{c}$ and $I_{c}$ is the critical current of a strip, $I_{c} = 2adJ_{c}$. Then

$$I_{dc} = d \int_{-a}^{a} J(u) \, du = d \int_{-\tilde{a}}^{\tilde{a}} \frac{\tilde{J}(\tilde{u})}{1 - (\pi \tilde{u}/D)^{2}} \, d\tilde{u}.
$$

(11)

Using Eq. (7) and the fact that $\tilde{H}(\bar{x}) = -\tilde{H}(-\bar{x})$ we obtain from Eq. (11)

$$I_{dc} = -2D \tilde{H}_{z}(\bar{x}) = D / \pi.
$$

(12)

Because $\bar{x} = D / \pi > \tilde{a}$, we derive from the last line of Eq. (8) using Eqs. (9) and (12)

$$b = \frac{D}{\pi} \arccosh \left( \frac{\cosh(\pi a/D)}{\cosh(\pi a_{dc}/Dl_{c})} \right).
$$

(13)

Fig. 2(a) shows the magnetic field distribution $H_{z}(x)/H_{d}$ (Eqs. (8) and (9)) for different values of $I_{dc}/I_{c}$ for a $z$-stack with $D/a = 0.5$ and for an isolated strip ($D/a \rightarrow \infty$). The field $H_{z}$ increases with decreasing $D/a$. In the limit $D \rightarrow d \rightarrow 0$ the magnetic field distribution $H_{z}(x)$ is that of an infinite slab of width $2a$ i.e.

$$H_{z}(x)/H_{d} = \begin{cases} 
0, & |x| < b', \\
-\frac{\pi}{d} \frac{x}{|x|} (|x| - b'), & b' < |x| < a, \\
-\frac{x}{|x|} \frac{\pi a I_{dc}}{d}, & |x| > a,
\end{cases}
$$

(14)

where $b' = a(1 - I_{dc}/I_{c})$.

Fig. 2(b) shows the current distribution $\tilde{J}(x)/I_{c}$ (Eq. (10)) for different values of $I_{dc}/I_{c}$ for a $z$-stack with $D/a = 0.5$ and for an isolated strip ($D/a \rightarrow \infty$). The smaller the vertical spacing $D$ becomes, the more the current density profiles resemble that of an infinite slab of width $2a$, i.e.

$$\tilde{J}(x)/I_{c} = \begin{cases} 
0, & |x| < b', \\
1, & b' < |x| < a, \\
0, & |x| > a.
\end{cases}
$$

(15)

2.2. Self-field hysteresis loss of the $z$-stack

When an ac current $I_{ac}(t) = I_{ac} \cos \omega t$ is flowing through each strip, the magnetic field and current density distributions can be expressed in terms of $H_{z}$ of Eq. (7) and (8). For the first half of the ac cycle ($0 < t < \pi / \omega$), the $z$-component of the magnetic field distribution $H_{z,1}(x,t)$ is given by [5]

$$H_{z,1}(x,t) = H_{z}(x,t) = I_{ac} J_{c} J_{dc} = I_{m} - I_{m}(t)
$$

(16)
and the current density distribution $\vec{J}(x,t)$ is given by
\[ \vec{J}(x,t) = \vec{J}(x,J_c, I_{dc} = I_m)
- \vec{J}(x,2J_c, I_{dc} = I_m - I_m(t)). \tag{17} \]

The self-field hysteresis loss $P/l$ per strip per length $l$ is defined as
\[ \frac{P}{l} = \frac{\omega d}{\pi} \int_0^{\pi/\omega} \left[ 2 \int_0^a E_x(u,t) \vec{J}(x,t) \, dx \right] \, du, \tag{18} \]
where $E_x(x,t)$ is the electrical field inside a strip generated by the self-field magnetic flux which moves in and out during each ac cycle. The integral over the time $t$ accounts for the time averaging of the loss.

The electric field $E_x$ is given by Faraday’s law as
\[ E_x(x,t) = \mu_0 \frac{d}{dt} \int_0^x H_z(u,t) \, du. \tag{19} \]

Using the fact that $\vec{J}(x) = J_c$ for those $x$ where $E_x(x,t) \neq 0$ one obtains
\[ \frac{P}{l} = -\frac{4\mu_0}{\pi} J_c \omega \int_{b(0)}^a \left[ \int_0^x H_z(u,t = 0) \, du \right] \, dx, \tag{20} \]
where
\[ b(0) = \frac{D}{\pi} \operatorname{arcosh} \left( \frac{\cosh(\pi a/D)}{\cosh(\pi a_{0mc}/D)} \right). \tag{21} \]

Eq. (20) can be simplified further by partial integration
\[ \frac{P}{l} = -\frac{4\mu_0}{\pi} J_c \omega \int_{b(0)}^a (x - a) H_z(x,t = 0) \, dx. \tag{22} \]

Using Eq. (6), Eq. (8) and Eq. (9) we obtain
\[ \frac{P}{l} = \frac{\mu_0}{\pi^2 \omega} \int_{b(0)}^a \left( a - x \right) \times \text{artanh} \left( \frac{\tanh^2(\pi x/D) - \tanh^2(\pi b(0)/D)}{\tanh^2(\pi a/D) - \tanh^2(\pi b(0)/D)} \right) \, dx. \tag{23} \]
Fig. 3 shows the self-field hysteresis loss \( P/l \) of the \( z \)-stack as a function of \( I_m/I_c \) for different values of \( D/a \) using \( \omega/2\pi = 50 \text{ Hz} \), \( a = 0.5 \text{ cm} \) and \( I_c = 100 \text{ A} \) and \( d = 1 \mu\text{m} \).

(isolated strip) Eq. (23) becomes the Norris formula [4], i.e.

\[
\frac{P}{l} = \frac{\mu_0}{2\pi} \frac{\omega I_c^2}{I_m} \left[ \ln \left( 1 - \frac{I_m}{I_c} \right) + \left( 1 + \frac{I_m}{I_c} \right) \ln \left( 1 + \frac{I_m}{I_c} \right) - \left( \frac{I_m}{I_c} \right)^2 \right].
\]  

(24)

where

\[
\frac{P}{l} = \frac{\mu_0}{12\pi^2} \frac{\omega I_c^2}{I_m^2} \text{ for } I_m \ll I_c
\]  

(25)

and

\[
\frac{P}{l} = \frac{\mu_0}{2\pi} \left( 2 \ln 2 - 1 \right) \omega I_c^2 \text{ for } I_m = I_c.
\]  

(26)

In contrast, for small \( D (D < a) \) the \( z \)-stack behaves similar to an infinite superconducting slab. In the limit \( D \rightarrow d \), the loss per strip corresponds to that of an infinite slab of width \( 2a \) which is, using Eq. (14), Eq. (16) and Eq. (22)

\[
\frac{P}{l} = \frac{\mu_0}{6\pi} \frac{a}{d} \frac{I_m^3}{I_c}.
\]  

(27)

In Fig. 3, the curve for \( D/a = d/a \) can be obtained by either using Eq. (23) or Eq. (27). As can be see in Fig. 3, \( P/l \) changes from a \( I_m^4 \) behaviour at large \( D/a \) to a \( I_m^3 \) behaviour at small \( D/a \).

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**Fig. 4.** \( x \)-array of infinitely long superconducting strips \((l \rightarrow \infty)\). The \( x \)-array periodicity is \( L \).
2.3. The $x$-array

Let us consider an infinite number of long superconducting strips periodically placed next to each other along the $x$-direction at a spacing $L$ where $L > 2a$. Such an $x$-array is shown in Fig. 4. From Eq. (1), assuming $a \gg d$, one finds

$$H_x(x) = -\frac{d}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-a}^{a} \frac{\tilde{J}(u)}{x - (u + nL)} \, du,$$

which alternatively can be expressed by [9,11]

$$H_x(x) = -\frac{d}{2L} \int_{-a}^{a} \tilde{J}(u) \cot(\pi(x - u)/L) \, du.$$

Introducing the variable transformations

$$\tilde{x} = \frac{L}{\pi} \tan\left(\frac{\pi x}{L}\right), \quad \tilde{u} = \frac{L}{\pi} \tan\left(\frac{\pi u}{L}\right)$$

and $\tilde{H}_x(x) = H_x(\tilde{x}) = -\frac{d}{2\pi} \int_{-\tilde{u}}^{\tilde{u}} \frac{\tilde{J}(\tilde{u})}{\tilde{x} - \tilde{u}} \, d\tilde{u},$

one obtains for Eq. (29)

$$H_x(x) = \frac{d}{2L} \int_{-a}^{a} \tilde{J}(\tilde{u}) \cot(\pi(x - u)/L) \, du,$$

which is the same as Eq. (7).

It is important to realize that the transformation $D \rightarrow iL$ (where $i = \sqrt{-1}$) causes Eqs. (5) and (6) to become Eqs. (29) and (30). This transformation has also been used in Ref. [9]. From Eq. (13), with $D \rightarrow iL$, we obtain for the position $b$ of the flux front in the strips

$$b = \frac{L}{\pi} \arccos\left(\frac{\cos(\pi a/L)}{\cos(\pi a/L_c/L)}\right).$$

Fig. 5(a) shows the magnetic field distribution $H_x(x)/H_d$ (Eqs. (8) and (9)) of the $x$-array for different currents $I_d/L_c$ with $L/a = 2.5$ as well as for the isolated strip ($L/a \rightarrow \infty$). The smaller the spacing $L$ is, the weaker $H_x(x)$ becomes, contrary to the $z$-stack behaviour. Fig. 5(b) shows the current density distribution $\tilde{J}(x)/J_c$ (Eq. (10)) for different values of $I_d/L_c$ for the $x$-array with $L/a = 2.5$ and

![Fig. 5. (a) $x$-array magnetic field $H_x/H_d$ and (b) current density $\tilde{J}/J_c$ versus $x/a$ at different values of $I_d/L_c$ for $L/a = 2.5$ (solid lines) and for an isolated strip ($L/a \rightarrow \infty$) (dashed lines).](image)
for an isolated strip. Away from the edges, \( \vec{J}(x) \) becomes nearly constant, i.e. independent of \( x \), when the spacing \( L \) is small. In the limit \( L \to 2a \) one obtains \( \vec{J}(x)/I_c = J_{dc}/I_c \).

### 2.4. Self-field hysteresis loss of the x-array

Using the transformation \( D \to iL \), one obtains for the self-field hysteresis loss \( P/l \) per strip per length \( l \) of the x-array

\[
P/l = \frac{\mu_0}{\pi} \frac{d I_m^3}{\omega a I_c} \int_{a}^{b} (a - x)
\]

\[
\times \text{artanh} \left( \frac{\tan^2(\pi x/L) - \tan^2(\pi b(0)/L)}{\tan^2(\pi a/L) - \tan^2(\pi b(0)/L)} \right) \text{d}x,
\]

(33)

where

\[
b(0) = \frac{L}{\pi} \arccos \left( \frac{\cos(\pi a/L)}{\cos(\pi a I_m/L_c)} \right).
\]

Fig. 6 shows the self-field hysteresis loss as a function of \( I_m/I_c \) for different spacings \( L/a \) using \( \omega/2\pi = 50 \text{ Hz} \), \( a = 0.5 \text{ cm} \) and \( I_c = 100 \text{ A} \). As can be seen, in the limit \( L \to 2a \) the loss \( P/l \) goes to zero while for \( L/a \to \infty \) Eq. (33) becomes the Norris formula [4] of Eq. (24).

It is important to realize that, for \( L \) close to \( 2a \), the loss which is due to the \( x \)-component of the magnetic field, \( H_x \), becomes more important than the loss caused by the perpendicular magnetic field \( H_z \) which has been considered so far. The loss from the \( x \)-component of the magnetic field is

\[
\frac{P}{l} = \frac{\mu_0}{24\pi} \frac{d I_m^3}{\omega a I_c},
\]

(35)

which is obtained from Eq. (27) by replacing \( d \) by \( 2a \) and \( a \) by \( d/2 \). \( P/l \) of Eq. (35) is shown as a dashed line in Fig. 6 where \( \omega/2\pi = 50 \text{ Hz} \), \( a = 0.5 \text{ cm} \), \( I_c = 100 \text{ A} \) and \( d = 1 \mu\text{m} \). The total loss cannot become smaller than the loss of Eq. (35).

### 3. Conclusion

Our investigations show that in the case of a \( z \)-stack made of long superconducting strips of rectangular cross-section, the \( z \)-component of the magnetic field inside the strip increases when the spacing \( D \) is decreased. For small \( D \) (\( D < a \)) the \( z \)-stack behaves magnetically similar to a superconducting slab. The \( z \)-stack self-field ac loss per strip is smallest for the isolated strip where \( P/l \sim I_m^3/I_c \) for \( I_m < I_c \). For \( D = d \) the loss is maximal and corresponds to that of a slab of width \( 2a \) where \( P/l \sim (a/d)(I_m^3/I_c) \).

In the case of the \( x \)-array, the \( z \)-component of the magnetic field inside the strip decreases with decreasing spacing \( L \). In the limit \( L \to 2a \) the \( z \)-component of the magnetic field vanishes and the current density distribution \( \vec{J}(x) \) becomes constant where...
The $x$-array self-field ac loss per strip is largest for the isolated strip and decreases with decreasing spacing $L$. For $L = 2a$ the loss from the $x$-component of the magnetic field, $H_x$, becomes larger than the loss from the $z$-component. The $H_x$-loss behaves like $P/l \sim (d/a)I_m/I_c$. The self-field loss of the $x$-array is always smaller than the self-field loss of the $z$-stack. The ratio between the minimum loss in the $x$-array, which occurs when $L = 2a$, and the maximum loss in the $z$-stack, which occurs when $D = d$, is $(d/2a)^2$. The behaviours of the $x$-array and the $z$-stack are reversed when the arrays are exposed to an ac magnetic field applied in the $z$-direction with zero applied currents [9]. In this case the loss of the $x$-array is always larger than the loss of the $z$-stack.

After this manuscript was completed we became aware of a preprint by Mawatari [13], which applies the same formalism to the self-field case. In contrast to his work our paper gives a more detailed derivation of the essential results and discusses the self-field ac loss in greater detail.

Acknowledgements

The author gratefully acknowledges the support of MM Cables, Metal Manufactures Ltd, the Energy Research and Development Corporation and DIST.

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