Surface instabilities and vortex transport in current-carrying superconductors

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We investigate the stability of the vortex configuration in thin superconducting films and layered Josephson-coupled superconductors under an applied current analytically and by numerical simulations of the time-dependent Ginzburg-Landau equation. We show that the stationary vortex lattice becomes unstable with respect to long-wavelength perturbations above some critical current $I_c$. We find that at currents slightly exceeding $I_c$, the vortex phase develops plastic flow, where large coherent pieces of the lattice are separated by lines of defects and slide with respect to each other. At elevated currents a transition to elastic flow is observed. We obtained the effective one-dimensional Ginzburg-Landau equation for a description of the vortex penetration from the edges. We discuss this transition in terms of a one-dimensional phase-slip phenomenon in superconducting wires with a periodically modulated temperature. We found several distinct dynamic vortex phases in the layered current-carrying superconductors. We show that for some intermediate range of the current, depending on the coupling between the layers, the coherent motion of the pancake vortices in different layers becomes unstable leading to dynamic decoupling. [S0163-1829(98)07205-1]

The transport in the vortex state of high-temperature superconductors (HTSC) attracts a great deal of current attention. Recent experimental and theoretical studies have revealed a rich variety of various vortex phases, including flux liquids, vortex lattices, glasses, dynamic resistive states, etc. These phases demonstrate novel dynamics such as low-temperature nonlinear glassy dynamics and thermally activated motion of pinned vortex liquid (see Ref. 1 for a review). Various transitions between different dynamic states (such as elastic and plastic flows) in a moving flux lattice have been predicted and observed both experimentally and by computer simulations.\textsuperscript{2-7} It has been found, in particular, that depinning of the vortex lattice is accompanied by plastic deformations, which play a primary role in formation of properties of the driven state. The question of the onset of plastic motion and its relation to specific mechanisms of pinning and intrinsic nonlinearity of the vortex state has become one of the central issues in the current study of vortex dynamics. Plastic effects are of special importance in high-temperature superconductors where processes of vortex cutting and reconnection are promoted by the stack-of-pancake structure of vortex lines in the highly anisotropic layered compounds involved.

Plastic effects in real finite samples are closely related not only to conditions of bulk pinning, but also to the process of flux penetration. The latter effects in particular can dominate vortex dynamics in technologically important superconducting thin strips. In this paper we address the problem of the onset of vortex motion in thin defect-free weakly coupled layered superconductors and the resulting bulk vortex dynamics. We demonstrate that the distinct driven vortex states can appear due to the intrinsic nonlinearity of the superconducting order parameter.\textsuperscript{5} We find analytically the threshold current $I_{c1}$ for vortex penetration though the surface and show that just above the threshold the distances between penetrating vortices are much larger than the vortex spacing in the bulk of the sample. Upon squeezing into the bulk of the lattice new vortices, coming from the surface, leave a trace in a form of chains of point defects. These chains separate large domains of an almost perfect vortex lattice, and the motion of the vortex lattice can be viewed as the sliding of these domains with respect to each other. Numerical simulations of the time-dependent Ginzburg-Landau equations (TDGLE) confirmed the proposed picture and revealed another remarkable feature of the vortex dynamics: the existence of the second critical current $I_{c2}>I_{c1}$ at which the plastic flow bifurcates to the elastic motion. In the elastic regime vortices penetrate the strip in a form of vortex rows with the period equal to the bulk vortex spacing. We propose a qualitative explanation of this phenomenon based on the analogy between a surface vortex-free layer of the strip and the one-dimensional (1D) superconducting wire with a spatial modulation of the critical temperature. We systematically derive a one-dimensional Ginzburg-Landau equation near the third critical field $H_{c3}$ and investigate the phase-slip phenomenon in this system. The generalization to the case of $H<H_{c2}$ is discussed. We study the effect of transport current on vortex alignment in the layered systems. We find that just above the threshold current at the onset of motion the perfect vertical alignment for vortices holds, this alignment is not destroyed by small fluctuations or by weak random pinning at moderate currents, although the in-plane coherence is not preserved. The resulting motion is similar to the plastic regime in two-dimensional superconductors.\textsuperscript{6} At higher currents, however, the perfect alignment of the vortices may become dynamically unstable, and vortices form dynamical decoupled state. This decoupling transition occurs even in the absence of thermal fluctuations as a function of the ap-
The depairing current density is

\[ j_p \]

where \( C \) are vector and scalar potentials, \( m \) are dimensionless variables assumes the form:

\[ \Psi = (\nabla - iA)^2 \Psi + (1 - |\Psi|^2) \Psi, \]

\[ j = |\Psi|^2 (\nabla \varphi - A) - (\nabla \mu + \partial \bar{A}), \quad \nabla j = 0, \quad \nabla A = 0, \]

\[ \Delta A = -\frac{1}{\lambda_{\text{eff}}} j \delta(z). \]

Here \( \Psi \) is the (complex) order parameter, \( \varphi = \arg \Psi \), \( A \) and \( \mu \) are vector and scalar potentials, \( j \) is the current density, and \( u \) is the dimensionless material parameter.\(^\text{11}\) The unit of length is the coherence length \( \xi \), the unit of time is \( t_0 = \xi^2 / Du \), where \( D \) is the diffusion constant, the field is measured in units of the upper critical field \( H_{c2} \). In these units the depairing current density is \( j_p = 2 / \sqrt{27} \approx 0.3875 \). The notion of a thin strip means \( s \ll d \ll \lambda_{\text{eff}} \), where \( s \) is the thickness and \( d \) is the width of the sample. The \( \delta(z) \) in Eq. (3) expresses the two-dimensional character of the problem. We consider an extreme type-II superconductor with \( \lambda_{\text{eff}} \gg 1 \) and, therefore, omit Eq. (3).\(^\text{12}\) We align the \( z \) axis with the applied magnetic field \( H \) perpendicular to the strip plane and the \( x \) axis parallel with the applied current (i.e., along the strip). The \( y \) axis traverses the strip and the strip edges are given by \((x,0)\) and \((x,d)\). We choose the Landau gauge for the vector potential, \( A = (Hy,0,0) \).

**B. Stationary current-carrying solution**

In an infinite type-II superconductor a mixed state exists in equilibrium in the form of a hexagonal lattice.\(^\text{13}\) In finite samples the vortex lattice is strongly deformed near the edges, where the flux-free layers are formed. For the magnetic field \( H \) not too close to \( H_{c2} \), the width of these vortex-free layers is of the order of the lattice spacing.\(^\text{14}\) The current distribution in the strip depends strongly on the dynamic status of the vortex lattice. The moving lattice generates almost uniform current distribution across the sample. If the lattice is at rest, the current is expelled to the narrow vortex-free layers at the edges of the sample. This stationary current-carrying vortex phase exists due to (surface) pinning of the vortex lattice. Above the critical current \( I_{c1} \) the stationary state loses its stability, vortices penetrate the sample, and the lattice begins to move. The motion of the lattice causes a redistribution of the transport current and profound voltage oscillations.

The determination of the critical current in a thin strip is, in general, a very complicated problem. The edge critical current in the strip can be found analytically at \( H = H_{c2} \) (in our units \( H_{c2} = 1 \)). In this case the magnitude of the order parameter \( |\Psi| \) inside the strip vanishes as \( \sqrt{H_{c2}} - H \). Near the edge the superconductivity disappears only at the third critical field \( H_{c3} \approx 1.69H_{c2} \) and \( |\Psi| \) remains finite at \( H = H_{c2} \). As a result the applied current flows mostly through the near-edge regions where superconductivity is not suppressed. Therefore, the contribution of the bulk vortex structure into formation of the edge critical current is negligible as \( H = H_{c2} \).

Let us first consider \( H_{c2} < H < H_{c3} \). The situation \( H < H_{c2} \) is considered later on. Since the superconductivity is localized near the edges, the edges can be treated independently if the width of the strip \( d \gg 1 \). The solution near the edge \((x,0)\) can be written in the form

\[ \Psi = F(y) \exp[ikx]. \]

Since the order parameter is practically zero in the bulk of the sample if \( d \gg 1 \), we can impose the following boundary conditions: \( \partial_y F = 0 \) and \( F(y) \to 0 \) for \( y \to \infty \). The parameter \( k \) is related to the total current \( I \) flowing near the edge through

\[ I = \int_0^\infty j_y dy = \int_0^\infty F^2(y)(k - Hy) dy. \]

The function \( F(y) \) obeys the following equation:

\[ \partial_y^2 F + F(1 - F^2 - (k - Hy)^2) = 0, \]

which Eq. (6) for \( H = H_{c3} \) can be solved analytically.\(^\text{13}\) Although a localized solution to Eq. (6) exists for arbitrary \( H \).
FIG. 2. The amplitude of the order parameter $F(y)$ for $H=1$, $k=0.39$ (solid line), and $k=1.32$ (dashed line).

$<H_3$, it is stable only in the range of fields $H_3<H<H_3$. For the fields $H 
eq H_3$ the solution is accessible only numerically. To do computations we used a differed difference algorithm with Newton iterations from the NAG library. The opposite edge of the strip is treated analogously. The (unstable) solution for $F$ near $H_{c3}$ is shown in Fig. 2. One sees that the order parameter is indeed strongly localized near the edge. The total current as a function of $k$ for $H=1$ is shown in Fig. 3 (we find that for $k<0$ the localized solution does not exist). The current $I(k)$ is bounded between two critical values $-I_1<I(k)<I_2$. The values $I_{1,2}$ grow as the field $H$ decreases (see Fig. 3). The apparent asymmetry $I_1 
eq I_2$ is caused by the magnetic field which breaks the $y ightarrow -y$ symmetry (in one case the current is parallel to the induced current, in the other case, antiparallel). For the strip infinite along $x$ the maximal possible supercurrent is equal to the sum of two currents flowing at the lower and the upper edges of the strip, i.e., $I=I_1+I_2$.

**C. Stability of stationary solution**

To examine the stability of Eq. (4) we seek a solution in the form $\Psi=[F(y)+\tilde{\xi}(x,y,t)]\exp(ikx)$ where $\tilde{\xi}(x,y,t)$ is a small perturbation. Linearizing the TDGLE with respect to $\tilde{\xi}$, splitting real and imaginary parts of $\tilde{\xi}=\tilde{a}+ib$, and representing the solution in the form $(\tilde{a},\tilde{b},\mu) \sim [a(y), b(y), \mu(y)] \exp(i\omega t+\omega t)+c.c.$, where $\omega$ is the perturbation wave number, one arrives at the eigenvalue problem for $\omega$:

$$
\begin{align*}
\omega a + (1-3F^2-(k-Hy)^2-q^2)a - 2qi(k-Hy)b &= 0, \\
\omega b + (1-F^2-(k-Hy)^2-q^2)b + 2qi(k-Hy)a &= 0, \\
\omega ub &= \mu - (q^2+uF^2)\mu.
\end{align*}
$$

The solution (4) becomes unstable when the negative $\omega(q)$ achieves zero for the first time (see Ref. 9). Since for $\omega=0$ the equation for $\mu$ has no localized solutions, the eigenvalue problem reduces to

$$
\begin{align*}
\partial_y^2 a + (1-3F^2-(k-Hy)^2-q^2)a &= 2qi(k-Hy)b, \\
\partial_y^2 b + (1-F^2-(k-Hy)^2-q^2)b &= -2qi(k-Hy)a.
\end{align*}
$$

For the most dangerous long-wavelength perturbations ($|q| \ll 1$) we will search for an approximate solution of Eqs. (8) in the form

$$
\begin{align*}
a &= a_0 + q(a_1 + q^2(a_2 + \cdots)), \\
b &= b_0 + q(b_1 + q^2(b_2 + \cdots)).
\end{align*}
$$

In the zeroth order in $q$ we obtain $a_0=0, b_0=F$ [compare Eqs. (6) and (8) at $q=0$]. The first order gives $b_1=0$ and the equation for $a_1$ as

$$
\partial_y^2 a_1 + [1-3F^2-(k-Hy)^2]a_1 = 2i(k-Hy)F.
$$

The solution to this equation is $a_1=i\partial_yF$ and is obtained by deriving Eq. (6) with respect to $k$. The second order in $q$ gives for $b_2$

$$
\partial_y^2 b_2 + [1-F^2-(k-Hy)^2]b_2 = F + 2(k-Hy)\partial_yF.
$$

Equation (11) has a bounded solution for $y \gg 1$ if the right-hand side (rhs) of Eq. (11) is orthogonal to the eigenfunction of the homogeneous problem. Equation (11) is self-adjoint with the eigenfunction given by $F$, and the solvability condition acquires the form

$$
\int_0^\infty F(F+2(k-Hy)\partial_yF)dy = \partial_y \int_0^\infty F^2(k-Hy)dy = 0.
$$

Since $I(k)=\int_0^\infty F^2(k-Hy)dy$ is a total current, the condition $\partial_y I(k)$ implies that the instability appears when the supercurrent achieves its extreme values $I_1$ or $I_2$. This long-wave instability is similar to that of 1D superconducting wires at the depairing current $I_p$.

**D. Correction from the vortex lattice for $H< H_{c2}$**

The above analysis holds also for $H< H_{c2}$ with the reservation that now the short-wave ($q \sim 1$) instabilities due to emerging Abrikosov vortices may develop. These instabilities, however, are controlled by the magnetic field and can be neglected when estimating critical currents for $H< H_{c2}$. Taking into account the formation of the vortex lattice in a
bulk of the sample at $H \leq H_{c2}$ the solution near the edge assumes the form $\Psi = \Psi_s + \Psi_b$ where $\Psi_s$ is the localized edge solution given by Eq. (4), and $\Psi_b$ describes a semi-infinite Abrikosov lattice (in the London limit $H \rightarrow 0$ some aspects of this problem were considered in Ref. 14)

$$\Psi_b = \sum_{n=0}^{\infty} C_n \exp\left[i k_n x - (y - k_n)^2/2\right].$$  \hspace{1cm} (12)

For a perfect hexagonal lattice $|C_0| = C_0 \sim \sqrt{H_{c2} - H} \ll 1$, $C_i = iC_{i+1}$, $k_i = k_0n + \sigma$, where $k_0 = \sqrt{3} \pi$, $\sigma$ is the distance from the end of the lattice to the edge of the strip. The distance $\sigma$ can be determined from the minimization of the energy. A simple estimate for $\sigma$ is obtained from the condition $\sqrt{\left\langle y^2 \right\rangle} = \max |\Psi_b| \sim C_0$ which gives

$$\sigma^2/2 \sim -\ln|C_0| = \frac{1}{2} \ln(H_{c2} - H).$$ \hspace{1cm} (13)

Using that for $H \rightarrow H_{c2}$, $\sigma \rightarrow \infty$, one immediately obtains for the part of the current $\delta l$ flowing in the lattice:

$$\delta l = \sum_{n=0}^{\infty} \int_0^\infty dy |C_0|^2 \exp\left[\left(y - k_n\right)^2\right] - |C_0|^2 \exp[-\sigma^2]$$

$$\sim (H_{c2} - H)^2 \rightarrow 0.$$

The boundary layer solution $\Psi_s$, completed by the lattice solution $\Psi_b$, is stable below the critical current. The estimate for the critical current differs from the one determined above only by a small quantity of order $(H_{c2} - H)^2$. The long-wave instability develops when the current achieves its critical values leading to the formation of a set of single zeros of $\Psi$ on the distances $2\pi/\sqrt{3}$ apart, which then serve as vortex nuclei. The current tears off these newly formed vortices from the surface and squeezes them into the lattice inside the strip. The wavelength of the instability is typically much larger than the bulk vortex spacing. This gives rise to formation of the lattice defects near the edges which can further develop into cracklike structures.

II. NUMERICAL RESULTS

To verify our predictions and to study further the nonlinear regimes of vortex motion we performed numerical simulations of the TDGLE at $H < H_{c2}$ at currents slightly above their critical value. To set an initial condition we generated a stationary vortex lattice at $I = 0$. We used the boundary conditions $\partial \Psi = 0$ (i.e., the boundary with vacuum) in the transverse direction and quasiperiodic conditions in the longitudinal direction $\left[\Psi(x+L,y) = \Psi(x,L)\right]$, $L$ is the strip length. We applied the quasispectral algorithm based on fast-Fourier transformation (see, for details, Refs. 9 and 10), the number of the mesh points was $1024 \times 256$ and the time step was $0.05 - 0.1$. The simulations were performed on the multiprocessor SGI Challenge computer. We identified the threshold of the surface instability by the appearance of a voltage through the sample and the onset of vortex motion. The wave number of the instability was deduced from the distribution of $\Psi(x,y)$. For $H = 0.9$ we obtained stability of the stationary state at the total current $I = 0.3075$, which is in excellent agreement with the analytic result $I_1 + I_2 = 0.31$. We have also found a small hysteresis for the onset of motion. Near the threshold the instabilities in the vortex pattern have a long-wave character and, therefore, the distances between the penetrating vortices significantly exceed the average bulk vortex spacing. The vortices penetrating through the surface break up the local order of the hexagonal lattice and create lattice defects. The domain of the coherent lattice confined by the two neighboring defects propagates into the bulk of the sample. The tracks of the confined defects form the defect chains separating the domains of a nearly perfect lattice. As a result the vortex configuration flows plastically via the motion of the large coherent pieces of the vortex lattice separated by the “cracks,” or channels, within which the lattice is destroyed. These channels cross the entire sample, allowing mutual sliding of the coherent domains (grains). The average size of a grain is about the wavelength of the most unstable surface mode. The observed plastic effect is very different from the plastic flow observed in the vortex system subject to a strong bulk pinning where the channels of the easy vortex motion bypassing strongly pinned islands develop under the applied current.

A. Transition from plastic flow to elastic flow

For higher currents (about 15–30% above the threshold) the transition to the regime where the lattice moves coherently is observed. In this current range the instability assumes a different character: new vortices penetrate sequentially along the surface forming a chain with vortex separations equal to the vortex spacing in the bulk, forming thus a new row of the bulk lattice (zipperlike penetration).

This transition is demonstrated in Fig. 5, where the slightly higher current than in the case shown in Fig. 4 is
applied. As we can see, the initially disordered vortex lattice reorders itself starting from the lower edge, where the vortices penetrate in an ordered manner. As a result, a perfect vortex lattice is formed. Remarkably, the close-packed direction of the lattice becomes aligned with the direction of the motion, whereas for the stationary lattice the close-packed direction is parallel to the edges. This effect of reorientation of the moving lattice is known in the context of superconductors with bulk pinning (see, e.g., Refs. 2, 17).

III. RELATION WITH THE PHASE-SLIP PHENOMENON

To obtain a better insight into the transition from plastic to elastic motion we note the analogy between the vortices penetrating from the edge of superconductor and phase slips in one-dimensional superconducting wire. Indeed, as we have found, the essential part of the current is concentrated near the edge (surface) of the sample, where the order parameter is not suppressed as $H\rightarrow H_{c2}$. It is plausible to neglect the transversal spatial dependence of the current and approximate the near-surface region by a one-dimensional wire. This assumption will be justified in the next section where we consider the amplitude equation for the surface instability at $H\rightarrow H_{c3}$. To account for dissipation effects due to bulk vortex motion, the wire is shunted by a parallel resistance. To include the periodic modulation of the width of the vortex-free layer by the bulk lattice, we introduce a periodic variation of the critical temperature along the wire.

The corresponding 1D TDGLE describing the wire is of the form

$$u(\partial_t + i\mu)\Psi = \partial_x^2\Psi + (\alpha(x) - |\Psi|^2)\Psi, I = I_s + I_r,$$

$$I_s = |\Psi|^2\nabla\varphi - \nabla\mu, I_r = \nu\frac{\mu(x+L) - \mu(x)}{L},$$

where $\nu$ is effective normal conductivity of the vortex lattice; $I_s$ is the current through the lattice; $\alpha = 1 + \beta \sin(k_0x)$ reflects the perturbation of the order parameter by lattice; and $\beta$ is the parameter of coupling which we choose to be exceedingly small, $\beta \sim 0.001$. We took the parameter $\nu$ of the order $20-40$, and the length of the wire equal to the length of the strip, and solved Eqs. (14) numerically. We started from the homogeneous distribution $\Psi = 1$ quenched by small-amplitude noise. This solution is stable upon the critical current $I_c = 2/\sqrt{27}$. We have observed spontaneous nucleation of the phase slips above $I_c$ (phase slips exist even below the critical current in the tiny range of the currents and do not nucleate spontaneously)$^{15}$, and while being close enough to the critical current, the distance between phase slips is found to be different from the lattice period [see Fig. 6(a)]. For elevated currents phase slips become coherent and nucleate simultaneously with exactly the period of the lattice [Fig. 6(b)]. This phase-slip array corresponds to the entrance of the row of vortices with the period of the lattice. As a result the lattice shifts as a whole, and the vortex lattice moves as an elastic medium. Remarkably, the perfect lattice starts to grow from the lower edge where the vortices penetrate the surface. Thus, the surface instability plays the organizing role in the formation of elastic motion.

A. Derivation of the amplitude equation for surface instability

In this section we obtain the one-dimensional amplitude equation for the description of the nonlinear stage of surface instability directly from the two-dimensional TDGLE in the vicinity of $H_{c3}$. We expect this equation to be qualitatively correct in the wider range of the fields (e.g., for $H \rightarrow H_{c2}$), although it cannot be derived systematically there.

We develop the perturbation theory starting from the stationary current-carrying solution in the vicinity of $H_{c3}$. The stationary edge solution is given by Eq. (4). The general form of the slowly varying time-dependent edge solution can be written as

$$\Psi = \int dk C_k(t) F_k(y) \exp[ikx],$$

where $C_k(t)$ is a function of time and the wave vector $k$. Now, taking into account that only functions with $k \neq k_0$ contribute to the integral (solutions corresponding to $k = k_0$...
decay rapidly for $H \rightarrow H_{c3}$, the above expression can be significantly simplified and reduced to

$$\Psi = \int dkC_k(t)F_k(y)\exp[ikx]$$

$$\approx \exp[ik_0x]\int dkC_{\delta k}(t)\exp[i\delta kjx]F_k(y) + \partial_kF_k(y)\delta k$$

$$+ \frac{1}{2}\partial_k^2F_k(y)(\delta j)^2.$$  \hspace{1cm} (16)

Now, using Eq. (16) we obtain the following expression for the wave function $\Psi$:

$$\Psi = \left(\frac{C(x,t,F_k(y)-i\partial_tC(x,t)\partial_kF_k(y)}{2} + \frac{\partial^2_kF_k(y)}{2} + \zeta\right)\exp[ikx],$$  \hspace{1cm} (17)

where $C(x,t) = \int dk\exp[i\delta kjx]C_{\delta k}(t)$ is a slowly varying function of $x,t$, $\zeta$ is a correction which we demand to be small. At $H \rightarrow H_{c3}$ the function $F$ can be expressed in terms of hypergeometrical functions $F((3-1)/4,1/2,H(y-k)^2).$ However, for practical purposes it is more convenient to approximate the solution by the function $F = \exp\left(-b\frac{Hy^2}{2}\right),$ \hspace{1cm} (18)

where $b = \sqrt{1-2/\pi}$ for zero applied current and $H_{c3} = 1/b$. This gives the answer that differs from the exact one by only 2%. The eigenvalue $\Lambda(k)$ of the linearized system as a function of $k$ is simply

$$\Lambda = 1 - \frac{Hb}{2} + 2k\sqrt{\frac{H}{\pi b} - (k-k_0)^2},$$

where $\epsilon = H_{c3} - H$ and $k_0 = H_{c3}/\sqrt{\pi} = 0.95.$

The condition that the function $C$ varies slowly can be satisfied only for $\epsilon \ll 1$. In order to find the equation governing the function $C$ we proceed with the asymptotic method. We obtain the evolution equation for $C$ from the orthogonality condition for function $\zeta$.

First, we rescale the variables

$$\mathcal{C} = \frac{C}{\sqrt{\epsilon}}, \quad X = \sqrt{\epsilon}x, \quad \mathcal{T} = \eta t, \quad \Lambda = \frac{\Lambda}{\epsilon}.$$  \hspace{1cm} (20)

Then, we represent $\zeta$ in the form of a series in $\epsilon$

$$\zeta = \sqrt{\epsilon}\xi_1 + \epsilon\xi_2 + \epsilon^{3/2}\xi_3 + \cdots.$$  \hspace{1cm} (21)

In first order we obtain linear equation $\mathcal{L}\xi_1 = 0$, where $\mathcal{L} = \partial^2_t - (Hy-k)^2$. Since the equation for $\xi_1$ has the trivial solution $\xi_1 = 0$, we proceed to the next order, and find

$$\mathcal{L}\xi_2 = 0.$$  \hspace{1cm} (22)

which also gives $\xi_3 = 0$. In the first nontrivial order of $\epsilon^{3/2}$ we obtain the following equation:

$$\mathcal{L}\xi_3 = \partial_t\mathcal{C}\mathcal{F} - \mathcal{X}\mathcal{C}\mathcal{F} + \frac{1}{2}\partial^2_t\mathcal{C}\mathcal{L}\partial^2_t\mathcal{F} - \partial^2_k\mathcal{C}\mathcal{F}$$

$$+ 2(Hy-k)\mathcal{C}_{XX}\mathcal{F} + |\mathcal{C}|^2\mathcal{F}^3 + i\mu\mathcal{C}\mathcal{F}.$$  \hspace{1cm} (23)

The solvability condition requires that the rhs of Eq. (23) is orthogonal to the kernel eigenfunction of $\mathcal{L}$. Since $\mathcal{L}$ is self-adjoint, this eigenfunction is simply $\mathcal{F}$. In order to simplify Eq. (23) it is useful to utilize the following relation: $\hat{L}F_{kk} = \Lambda_{kk}F + 4(k-Hy)\partial_k\mathcal{F} + 2F\partial^2_k\mathcal{F}$. Thus, performing the integration, we obtain the following equation:

$$\partial_t\mathcal{C} = \mathcal{X}\mathcal{C} - \frac{1}{2}\partial_k^2\mathcal{C} - \partial^2_k\mathcal{C} - i\mu\mathcal{C} - \alpha|\mathcal{C}|^2\mathcal{C},$$

where $\alpha = \int_0^\infty x^2dy/\int_0^\infty F^2dy$. Using approximations for $F_{kk}$ Eqs. (19,18) we obtain the following: $\alpha = 1/\sqrt{2}, \frac{1}{2}\partial_k^2\Lambda = 1$.

The electric potential $\mu$ in the first order is given by the equation $\nabla^2\mu = 0$. It can be chosen in the form $\mu = -Ex$, where $E$ is the applied electric field (in the first order the applied field is not perturbed by the supercurrent because the order parameter $\Psi$ vanishes at $H_{c3}$. Performing obvious scaling $T \rightarrow T/\sqrt{\Lambda}, X \rightarrow X/\sqrt{\Lambda}, \Psi = \mathcal{C}/\sqrt{\Lambda}\alpha, E = E/\sqrt{\Lambda}$, we obtain the following one-dimensional TDGLE

$$\partial_t\psi = \psi + \partial^2_t\psi - |\psi|^2\psi + iEX\psi.$$  \hspace{1cm} (25)

Equation (25) coincides with the Ginzburg-Landau equation for a one-dimensional superconducting wire. Although we obtain Eq. (25) close to the third critical field $H_{c3}$, we expect that it will be qualitatively correct also in the vicinity of $H_{c2}$. Thus, it justifies our description of the vortex penetration in terms of one-dimensional phase-slip phenomenon.

We expect that Eq. (25) is qualitatively correct also for $H < H_{c2}$ since it preserves principal features of the original TDGLE, namely, gauge invariance, translation symmetry, etc. However, the nonlinearity may assume a more complicated form, e.g., $g(|\psi|^2)\psi$, where $g$ is some monotonous function. Also, for $H < H_{c2}$ the vortex lattice in a bulk of the sample introduces periodic in space modulation of the order parameter.

B. Periodic lattices of phase slips in one-dimensional TDGLE

We investigated Eq. (25) both numerically and analytically. Since Eq. (25) is significantly simpler than the original TDGLE, it is possible to obtain analytical results on dynamic behavior. Our approach differs from that of Ref. 16, where the lattices of phase slips were generated by strong defects. In Fig. 7(a) we show the spatiotemporal diagram of $|\psi|$. The black dots correspond to the zero of $|\psi|$, and, therefore, to the phase-slip events. After a long transient (not shown on the figure) the phase slips form a triangular lattice, which resembles the vortex lattice in the original two-dimensional TDGLE. However, there is a significant difference. The periods of the phase-slip lattice along $X,T$ are not uniquely selected by the applied field $E$, but vary continuously within some band. We conclude that the periods of the phase-slip lattice are determined by external conditions, such as the size.
of domain, initial $\psi$ distribution, and, more importantly, by
the periodic modulation (not included in the model) due to
vortex lattice in the bulk.

The continuous behavior of the temporal $\tau$ and spatial $L$
periods is demonstrated in Fig. 8. The dependence of $\tau$ on $E$
is given exactly by the expression

$$\tau L = \frac{4\pi}{E},$$

(26)

which is obviously a direct consequence of the gauge invariance
of Eq. (25).

As one can see from Fig. 9 the supercurrent $j_s = \text{Im} \psi^* \psi_X$ decays as the electric field grows. This does not
contradict Ohm’s law since the total current (supercurrent +
normal current) grows. The phase-slip lattice with the pe-
riod $L$ exists up to critical value $E_c$ of the electric field. At
$E > E_c$, two distinct scenarios were observed: (a) the period $L$
decreases, resulting in jumps in the current-voltage character-
istic; (b) the solution vanishes ($\psi \rightarrow 0$) when $E$ becomes
too high. On the contrary, a decrease of $E$ leads to the break-
down of the solution with the period $L$ in favor of solution
with larger $L$. Remarkably, the transition occurs through
spontaneous formation of the lattice of spatiotemporal dislo-
cations [see Fig. 7(b)].

The quantitative description of the phase-slip pheno-
menon can be achieved in the limit when both $L$ and $\tau$ are
simultaneously large, i.e., $E \ll 1$. In this limit an analog of the
London approximation can be developed. For this purpose it
is convenient to consider Eq. (25) for the variable $\bar{\psi} =
\psi \exp[-i\text{EXT}]$. Then Eq. (25) reduces to

$$\partial_T \bar{\psi} + (\partial_X + iET)^2 \bar{\psi} - |\bar{\psi}|^2 \bar{\psi}.$$  (27)

Introducing amplitude $I = |\psi|$ and phase $\varphi = \text{arg} \bar{\psi}$ variables
one reduces Eq. (27) to

$$\partial_T I = \partial_X^2 I + I - I^3 - (\partial_X \varphi + ET)^2 I,$$  (28)

$$\partial_T \varphi = \partial_X^2 \varphi + (\partial_X \varphi + ET) \frac{\partial_X I^2}{I^2}.$$  (29)

As we can see from Fig. 7(a) the amplitude of the order
parameter $I$ is practically $X$ independent between the phase-
slip event and is only a function of time $T$. This circumstance
simplifies the analysis significantly. Assuming that $I$ is the
function of time $T$ only, one finds that the equation for phase
(29) becomes simply the diffusion equation

$$\partial_T \varphi = \partial_X^2 \varphi$$  (30)

and the equation for the amplitude of $\psi$ reduces to

$$\partial_T I = I - I^3 - (ET)^2 I.$$  (31)

Equation (31) has an analytic solution
where arbitrary constants $C, T_0$ will be defined later on.

The equation for the phase $\phi$ is linear, and, therefore, the solution can be sought as a superposition of isolated phase slips. An isolated singular phase-slip event means that a $2\pi$ phase gain occurs upon encircling the phase-slip center in the $T,X$ plane. Such a solution to the one-dimensional diffusion equation is of the form

$$
\phi = \phi_0(X/\sqrt{4T})
$$

$$
= \begin{cases} 
\pm 2\sqrt{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{T}} d\exp[-s^2] & \text{for } T>0 \\
0 & \text{for } T<0.
\end{cases}
$$

The function $\phi_0$, related to the erfc function, experiences a $2\pi$ jump when the line $T=0$ is crossed for $X>0$. A periodic solution in the form of a staggered (displaced hexagonal) array of individual phase slips from Eq. (33) is

$$
\phi = 2\sqrt{\pi} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[ \phi_0 \left( \frac{X-Ln}{2\sqrt{T}-\tau m} \right) + \phi_0 \left( \frac{X-L(n+1/2)-\sigma}{2\sqrt{T}-\tau m - \tau/2} \right) \right].
$$

For $\sigma=L/2$ one obtains a perfect hexagonal lattice and $\sigma=0$, $L$ leads to a square array. The stability of the solution (34) can be examined by consideration of the formal free-energy functional associated with Eq. (30): $F=\frac{1}{2} \int dX (\partial_X \phi)^2$. Substituting expression (34) into the free-energy functional and keeping only the terms which depend on $\sigma$ (obviously, $\sigma$-independent terms do not affect the stability), we arrive at the relation

$$
F_i = \frac{1}{2} E \sqrt{\pi S} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{2T - \tau m - \tau/2}} \times \exp \left[ - \frac{(L(n_1 - n_2 - 1/2) + \sigma)^2}{4(2T - \tau (m_1 - m_2) - \tau/2)} \right].
$$

The periodicity of $I^2$ can be enforced by adjusting the constants $C_0, T_0$ in Eq. (32). In the time interval $0 \leq T < \tau/2$ it is possible to take $T_0=0$ and to find $C_0$ from the periodicity condition $I(0)=I(\tau/2)$. It yields

$$
C_0 = -\exp(\tau - E^2 \tau/12) - 1.
$$

Finally, we obtain the time-averaged supercurrent

$$
\langle J_s \rangle = \frac{1}{L \tau} \int_0^L dx J_s
$$

$$
= \frac{2E \int_0^{\tau/2} dt \int_0^L dx J_s}{C_0 + 2 \int_0^\tau \exp[2s - 2E^2 s^3/3]}.
$$

The dependence of $\langle J_s \rangle$ in $E$ is shown in Fig. 8. For comparison the numerically obtained values of $\langle J_s \rangle$ are shown for the same parameter. We see that although the analytical results are very close to numerical results (and even coincide at some points), there is yet a systematic discrepancy; the theoretical curve grows with $E$ (although very slow), while the numerically obtained $J_s(E)$ decays with the increasing $E$. We conclude here that the spatial dependence must be included to fix the discrepancy.

To conclude this part, we review our previous result on the transition from plastic to elastic motion. We have found on the basis of the amplitude equation (25) that the periodic phase-slip solution with the period $L$ exists up to some critical value of the electric field $E$. The solution becomes unstable in favor of the lattice with larger period $L$ if $E$ decreases below some critical value. Since at the threshold of
the instability we have $E \rightarrow 0$, it implies that $L \rightarrow \infty$. Therefore, at the threshold the period of the phase-slip lattice exceeds significantly the bulk lattice spacing leading to the plastic flow regime. However at sufficiently large $E$ (or applied current) the phase-slip lattice with the period close to the bulk vortex spacing becomes stable. The modulation of the order parameter due to bulk vortex lattice forces these two periods to coincide. This corresponds to the transition to elastic flow. It is the exact match between the hexagonal array of the phase slips in the edge layer and the moving hexagonal vortex lattice that provides the elastic motion of the vortex configuration.

**IV. SURFACE INSTABILITY IN LAYERED SUPERCONDUCTORS**

Now we consider generalization of our results for the layered system. We proceed with the time-dependent Ginzburg-Landau-Lawrence-Doniach model for the layered superconductors, which in dimensionless variables is of form (we rescale here $t \rightarrow ut$):

$$\dot{\psi}_l + i \mu_0 \psi_l = (\nabla - i A)^2 \psi_l + (1 - |\psi_l|^2) \psi_l$$

$$+ \frac{1}{\gamma^2} (\nabla \psi_{l+1} + \nabla \psi_{l-1} - 2 \nabla I).$$
(40)

The current density within the layer $J_l$ (i.e., parallel to the surface) is given by

$$J_l = |\psi_l|^2 (\nabla \varphi_l - A) - (\nabla \mu_l/u + \partial \varphi_l).$$
(41)

The Josephson current between the $l+1$ and $l$ layers (perpendicular to the surface) $J_l$ is of the form

$$J_l = \frac{1}{\gamma^2} [\text{Im}(\psi^*_{l+1} \psi_{l+1} - (\mu_{l+1} - \mu_l)].$$
(42)

The current conservation condition is of the form

$$\text{div} J_l + J_{l-1} - J_{l+1} = 0.$$  
(43)

We focus on pure Josephson coupling between the layers. The coupling is characterized by the anisotropy parameter $\gamma$. In perspective high-$T_c$ materials, such as YBCO, BSCCO, the coupling between the layers is weak and $\gamma$ is very large. (For YBCO $\gamma \approx 7-9$ and for BSCCO $\gamma \approx 40-50$). A description of the interlayer interaction as the pure Josephson coupling (neglecting the magnetic interactions) is a reasonable approximation with the reservation that the sample is not too thick, i.e., the number of layers does not exceed 40–50.

Pancake vortices are perfectly aligned with the applied magnetic field in clean (defect-free) multilayered type-II superconductors at equilibrium. This means that $\psi_l$ does not depend on $l$. Perfect alignment persists also for a driven vortex lattice since the homogeneous (defect-free) TDGLE is translationally invariant. However, it does not necessarily imply a stability of driven aligned (or synchronous) state in the range of applied currents. We show by numerical simulations of TDGLE and qualitative arguments that the perfect order of vortex motion can be destroyed by specific instabilities, arising from the noncoherent vortex penetration from the sample edges. This effect is a manifestation of the surface instability of the two-dimensional system in a quasi-three-dimensional situation. In two-dimensional samples this instability is responsible for the disintegration of perfect hexagonal order of the driven vortex lattice and leads to plastic deformations. In three dimensions and in layered systems the instability leads to disintegration of the aligned stacks of pancake vortices and results in dynamic entangled or decoupled state. This dynamic state is characterized by the significant suppression of the interlayer correlations and can be viewed as a three-dimensional plastic flow.

The dynamic entangled state emerges if the effective intensity of the (dynamic) fluctuations caused by the surface instability exceeds the competing coupling interaction. The dynamic fluctuations can be characterized by the corresponding Lyapunov exponents $\Lambda_m$ (see, e.g., Ref. 18). The Lyapunov exponents are zero or negative for stable periodic motion (corresponding, e.g., to elastic flow) and are positive for plastic flow, characterized also by the chaotic voltage response.10,8 Chaotic oscillations of the voltage occur at the threshold of vortex penetration.10,8 As a result, one may expect that there is a critical coupling, characterized by $\gamma_c$, below which the aligned state is unstable. In order to see that it is useful to consider TDGLE, linearized about the $l$-independent current-carrying solution $\Psi_l = \Psi_0(x,y,t) \mu_0 = \mu_0(x,y,t)$ for all $l$. Representing the perturbative solution in the form $\Psi_l = \Psi_0 + \eta_l$ and $\eta_l = \eta_l + \nu_l$, where $\eta_l$ and $\nu_l$ are small perturbations, we obtain the linear equation for the order parameter

$$\dot{\eta}_l + i u \mu_0 \eta_l + i u \Psi_0 \nu_l =$$

$$(\nabla - i A)^2 \eta_l + \eta_l - 2 |\Psi_0|^2 \eta_l - \Psi_0^* \eta_l$$

$$+ \frac{1}{\gamma^2} (\eta_{l+1} + \eta_{l-1} - 2 \eta_l),$$
(44)

and the corresponding linear equation for $\nu_l$. Since all the coefficients in Eqs. (44) are $l$ independent, it is possible to apply the discrete Fourier transformation with respect to the index $l$: $(\tilde{\eta}_k, \tilde{\nu}_k) = \sum_{l=1}^{N} (\eta_l, \nu_l) \exp[-2\pi ik l/N]$. The transformed equation assumes the following form:

$$(\dot{\eta}_k + i u \mu_0 \eta_k + i u \Psi_0 \nu_k = (\nabla - i A)^2 \eta_k + \eta_k - 2 |\Psi_0|^2 \eta_k$$

$$- \Psi_0^* \eta_k - \frac{4 \sin^2 k/2}{\gamma^2} \eta_k.$$  
(45)

The aligned state is stable, if for all $k \neq 0$ the Lyapunov exponents are negative. The $k = 0$ corresponds to the Lyapunov exponent of the aligned state and can be positive in the onset of surface instability. However, this exponent is not responsible for the breakdown of the alignment since it corresponds to $l$-independent fluctuations. The spectrum of Lyapunov exponents $\Lambda_k$ can be found analytically in the limit of $u \rightarrow 0$. We have immediately

$$\Lambda_k = \Lambda_0 - \frac{4 \sin^2 k/2}{\gamma^2}, \quad k = \frac{\pi m}{N}, \quad m = 0,1, \ldots , N-1.$$  
(46)
From Eq. (46) we see that the aligned state is stable if the coupling between the layers $1/\gamma^2$ exceeds some critical value $\gamma_c$, defined by the condition $\Lambda_0 - 4 \sin^2 \pi N/\gamma^2 = 0$. For any $\gamma > \gamma_c$, the aligned state is unstable.

For an arbitrary $u \neq 0$ the analytical expression for Lyapunov exponents in terms of $\Lambda_0, \gamma$ is not known at present. However, we expect that the dependence, similar to Eq. (46) also holds for $u \neq 0$. Accurate values for $\gamma_c$ can be obtained with the aid of numerical simulations of TDGLE. Since the simulation of the multilayered system is a very time-consuming problem we restrict ourselves to a system consisting of two identical layers ($N=2$). We fix the total current flowing in both layers, and in order to set an initial condition we generate identical vortex configurations in both layers and then slightly perturb them. We use the superconductor-vacuum boundary conditions in the lateral layers and then slightly perturb them. We use the condition we generate identical vortex configurations in both current flowing in both layers, and in order to set an initial

The perfect alignment corresponds to $W=1$ and $W$ decreases as the alignment is destroyed. As a measure of alignment we use the time-averaged correlation coefficient between the layers: $\kappa = \langle W \rangle = \langle \cos(\phi_1 - \phi_2) \rangle$.

Selected results are shown in Figs. 10 and 11. The dependence of the critical coupling $\gamma_c$ on the current is shown in Fig. 10. The amplitude of the order parameter and the distribution of $W$ in the regime of dynamic decoupling is shown in Fig. 11. As one can see from the figure, the desynchronization occurs primarily at the sample edges and then propagates inside the sample.

The results of the simulations can be briefly summarized as follows. At the onset of vortex motion the aligned (synchronous) state is stable even for very weak coupling. It can be easily understood if we note that the Lyapunov exponent becomes nonzero and grows further as soon as vortices start to slide. At the threshold arbitrary weak coupling is sufficient for the synchronization. The motion of the vortices just after its onset is however plasticlike, where large coherent pieces of the perfect lattice are separated by cracklike lines of lattice defects. This regime is similar to that of the single-layered system. At higher currents the synchronous motion sets, but this regime persists only up to some critical current $j_{c,1}$ where the perfect alignment breaks down and the decoupling transition takes place. Upon further increasing of the current the re-entrance of the perfect alignment is observed at $j = j_{c,2} > j_{c,1}$. This effect is related to the transition from the plastic to the elastic regime of vortex motion.

CONCLUSION

We have investigated vortex transport and stability properties of the vortex configuration in thin superconducting strips. Our interest was driven first by the importance of thin films and strips for possible microelectronic applications, and second by the fact that thin strips serve as a convenient model system for the study of the pinning-stimulated plastic effects in vortex dynamics. We have shown that the stationary vortex lattice becomes unstable with respect to long-wavelength perturbations above some critical current $I_c$. We have found that at currents slightly exceeding $I_c$ the regime of plastic flow develops, where large coherent pieces of the lattice are separated by cracklike lines of defects and slide with respect to each other. At further increasing current a
transition to elastic flow is observed. We have derived the effective one-dimensional Ginzburg-Landau equation for the description of the vortex penetration from the edges. We have shown that for some intermediate range of currents, depending on the coupling between the layers, the coherent motion of the pancake vortices in different layers becomes unstable giving rise to dynamic decoupling. The discussed results were obtained for clean strips in the absence of bulk pinning. In the presence of bulk pinning a peculiar competition between the two pinning mechanisms may take place: in some range of the currents (depending on the bulk pinning strength) the lattice defects generated at the surface can anneal within the bulk due to random collisions with the pinning sites. As a result, the lattice may restore the perfect hexagonal structure in the bulk of the strip even if the numerous defects are produced at the edges. This transition promotes also alignment between the layers.

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