DIPOLES IN WHOLE AND HALF-SPACES AND TILT EFFECTS

We will in the following consider a vertical current dipole in an infinite medium and then use this result to find the solution for an infinite half space.

Diffusion equation for inductive currents

We consider the injected current flux as composed by a source term $\mathbf{J^e}$ localized to the transmitter. It is important to note that $\mathbf{J^e}$ is a discontinuous field which is zero outside the transmitter. The electric field $\mathbf{E}$ is valid everywhere, also inside the transmitter. Since the conductance of the transmitter σ^e is much larger than the conductance σ in the external medium (The ratio between the two is larger than 10^6), the electric field inside the transmitter will satisfy

$$\sigma \mathbf{E} << \sigma^e \mathbf{E} = \mathbf{J^e} \ . \tag{1}$$

We can therefor write

$$\mathbf{J} \approx \mathbf{J^e} + \sigma \mathbf{E} \ , \tag{2}$$

which is a good approximation inside the transmitter and exact outside the transmitter. Neglecting the displacement current Maxwell's equations on differential form are

$$\nabla \cdot \mathbf{D} = \rho \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{5}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ . \tag{6}$$

Applying $\nabla\cdot$ on Eq.(5) gives that

$$\nabla \cdot \mathbf{J} = 0 \ . \tag{7}$$

We further apply $\nabla\times$ on Eq.(5), and use the vector identity

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} \tag{8}$$

and Eq.(4), Eq.(2) and Eq.(6) which gives

$$-\nabla^2 \mathbf{H} = \nabla \times \mathbf{J}^e + \sigma \nabla \times \mathbf{E} = \nabla \times \mathbf{J}^e - \mu\sigma \frac{\partial \mathbf{H}}{\partial t} \ . \tag{9}$$

If we apply $\nabla\times$ one more time one this equation and use Eq.(5) we get

$$-\nabla^2 \mathbf{J} = \nabla \times \nabla \times \mathbf{J}^e - \mu\sigma \frac{\partial \mathbf{J}}{\partial t} \ , \tag{10}$$

which is a diffusion equation for the current flux $\mathbf{J}$.

$$\frac{\partial \mathbf{J}}{\partial t} = D\nabla^2 \mathbf{J} + s(\mathbf{r}, t) \ , \tag{11}$$

where the diffusion constant $D = 1/(\mu\sigma)$, and the source term is given by

$$s(\mathbf{r}, t) = D\nabla \times \nabla \times \mathbf{J}^e \ , \tag{12}$$

From the observation that $\mathbf{J}$ satisfies a diffusion equation, it is possible to obtain a rough qualitative understanding of how the current, or E-field, evolves after the dipole is turned off. Any localized perturbation in the current field will spread diffusively. This means that, a perturbation will only be felt at some distance r after a time t that is given by $r^2 = 2Dt$. In particular, if we suddenly turn a dipole field off, we will only see a distant effect after some time. Since the diffusion equation is linear, turning the dipole off is equivalent to adding another dipole that cancels the original one. This second dipole has to have equal and opposite strength. This is illustrated in Fig. 1. The effect of the opposite dipole will be localized within the distance $r(t) = \sqrt{2Dt}$, and since it carries the same overall current, it will locally dominate the original dipole field and cause a sign change in the resulting current density (Fig. 1 c).

Equation (10) has the general solution

$$\mathbf{J}(\mathbf{r}, t) = \int d^3\mathbf{r}' dt' G(\mathbf{r}-\mathbf{r}', t-t') \mathbf{s}(\mathbf{r}', t') = D\int d^3\mathbf{r}' dt' G(\mathbf{r}-\mathbf{r}', t-t') \nabla\times\nabla\times\mathbf{J}^e(\mathbf{r}', t') \ . \tag{13}$$

The j'th component of $\nabla \times \mathbf{F}$ where $\mathbf{F}$ is a general vector function can be written as

$$(\nabla \times \mathbf{F})_j = \epsilon_{jkl} \frac{\partial F_l}{\partial x_k} \ . \tag{14}$$

Here ϵ_{jkl} is equal to 1 when jkl is a cyclic permutation of 123 and equal to -1 when jkl is not a cyclic permutattion. It follows that ϵ_{jkl} equal to zero if two of jkl are equal. In Eq.(14) Einstein's convention is used. It means that two consecutive equal indexes implies summation, i.e. $a_i b_i = \sum_i a_i b_i$. Consider the expression

$$\int_{-\infty}^{\infty} d^3\mathbf{r}' G(\mathbf{r}-\mathbf{r}', t-t') \epsilon_{jkl} \frac{\partial J_l^e}{\partial x_k'} = -\int_{-\infty}^{\infty} d^3\mathbf{r}' \epsilon_{jkl} \frac{\partial G(\mathbf{r}-\mathbf{r}', t-t')}{\partial x_k'} J_l^e = +\int_{-\infty}^{\infty} d^3\mathbf{r}' \epsilon_{jkl} \frac{\partial G(\mathbf{r}-\mathbf{r}', t-t')}{\partial x_k} J_l^e \tag{15}$$

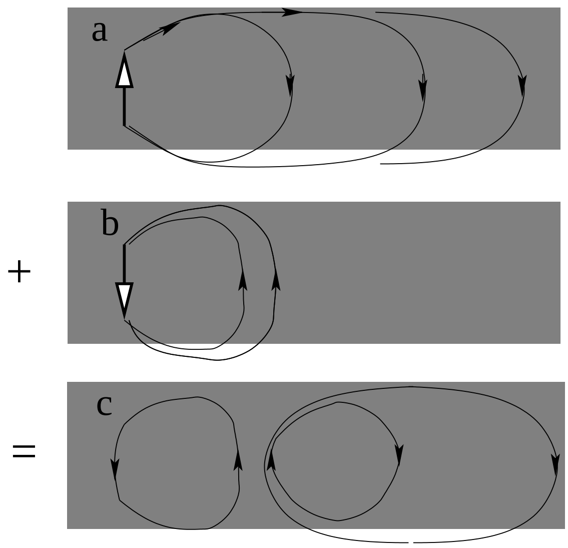

Figure 1: Current field around current dipoles, shown as open arrows, in a sea, shown as gray. The current field after the dipole is turned off (c) may be considered the superposition of a stationary dipole field (a) and an evolving dipole field that is caused by an opposite dipole source (b), which is suddenly turned on.

where we have performed a partial integration and used that the Green's function goes to zero at large distances. We have further used that the derivative $\frac{\partial G}{\partial x'_k} = -\frac{\partial G}{\partial x_k}$. This implies that the $\nabla \times \nabla \times$ operator can be brought outside the integral of Eq.(13) and we get

$$\mathbf{J}(\mathbf{r}, t) = D\nabla \times \nabla \times \int d^3\mathbf{r}' dt' G(\mathbf{r} - \mathbf{r}', t - t')\mathbf{J}^e(\mathbf{r}', t') \ , \tag{16}$$

where $\nabla \times \nabla \times$ operates on $\mathbf{r}$ and not on $\mathbf{r}'$. Let us assume that source current is constant up to the time $t = 0$ where it is turned off $\mathbf{J}^e(\mathbf{r}, t) = \mathbf{J}^e(\mathbf{r})\Theta(-t)$, and that we consider the response $\mathbf{J}(\mathbf{r}, t)$. The Greens function for the diffusion equation with a source term was shown earlier to be

$$G(\mathbf{r}, t) = \frac{e^{-\frac{\mathbf{r}^2}{4Dt}}}{(4\pi D t)^{3/2}}\Theta(t) \ . \tag{17}$$

Inserting this into the Eq.(16) gives

$$\mathbf{J}(\mathbf{r}, t) = D\nabla \times \nabla \times \int d^3\mathbf{r}' dt' \frac{e^{-\frac{(\mathbf{r}-\mathbf{r}')^2}{4D(t-t')}}}{(4\pi D(t - t'))^{3/2}}\mathbf{J}^e(\mathbf{r}', t') \ . \tag{18}$$

We will now assume that we have a small dipole compared to distance from the dipole to the receiver, $r' << r$. We will further use the definition of the current dipole moment $\mathbf{p}$

$$\mathbf{p} \equiv \int d^3\mathbf{r}' \mathbf{J}^e(\mathbf{r}') \ , \tag{19}$$

which gives

$$\mathbf{J}(\mathbf{r},t) = D\nabla\times\nabla\times\left(\mathbf{p}\int_{-\infty}^{0} dt' \frac{e^{-\frac{\mathbf{r}^2}{4D(t-t')}}}{(4\pi D(t-t'))^{3/2}}\right) . \tag{20}$$

Notice that the upper integration limit $t' = 0$ reflects the fact that the current dipole is turned off at this time. By changing variables $u^2 = \mathbf{r}^2/(4D(t-t'))$ and $dt' = \mathbf{r}^2 du/(2Du^3)$ this integral can be expressed by the well known error function

$$\mathbf{J} = \nabla\times\nabla\times\left(\mathbf{p}\int_{0}^{\frac{r}{\sqrt{4Dt}}} du \frac{e^{-u^2}}{2\pi^{3/2}r}\right) = \nabla\times\nabla\times\left(\mathbf{p}\frac{\mathrm{erf}(r/(4Dt)^{1/2})}{4\pi r}\right) , \tag{21}$$

where the error function is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt . \tag{22}$$

The error function is defined so that $\mathrm{erf}(x) \to 1$ when $x \to \infty$. We will also use the definition

$$E(x) = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) . \tag{23}$$

Using the identity of Eq. (**??**) and introducing the definitions $\alpha = 1/\sqrt{4Dt}$ and

$$f(r) = \frac{\mathrm{erf}(\alpha r)}{4\pi r} \tag{24}$$

we can write the current density as

$$\mathbf{J} = [\nabla(\mathbf{p}\cdot\nabla f(r)) - \mathbf{p}\nabla^2 f(r)] = (\nabla(p_r f'(r)) - \mathbf{p}\nabla^2 f(r)) \tag{25}$$

where, in the last expression, we have made use of the fact that $f(r)$ has spherical symmetry. In spherical coordinates the unit vectors have the form

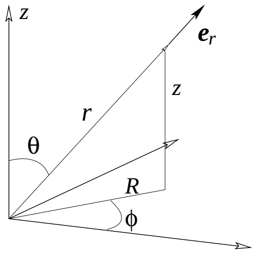

Figure 2: Basic definitions for spherical coordinates (r, Θ, ϕ) and cylindrical coordinates (R, z, ϕ).

$$\begin{aligned}\mathbf{e}_r &= [\sin\Theta\cos\phi, \sin\Theta\sin\phi, \cos\Theta] \\ \mathbf{e}_\Theta &= [\cos\Theta\cos\phi, \cos\Theta\sin\phi, -\sin\Theta] \\ \mathbf{e}_\phi &= [-\sin\phi, \cos\phi, 0] \end{aligned} \tag{26}$$

and the gradient operator may be written

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\Theta}{r}\frac{\partial}{\partial \Theta} + \frac{\mathbf{e}_\phi}{r\sin\Theta}\frac{\partial}{\partial\phi} \; . \tag{27}$$

Since $p_r = \mathbf{e}_r \cdot \mathbf{p}$ where $\mathbf{p}$ is just a constant vector, it is easy to demonstrate that

$$\frac{\partial p_r}{\partial \Theta} = p_\Theta \text{ and } \frac{1}{\sin\Theta}\frac{\partial p_r}{\partial \phi} = p_\phi \; . \tag{28}$$

It follows from this that

$$\nabla(\mathbf{p}\cdot\nabla f(r)) = \mathbf{e}_r p_r f''(r) + (p_\Theta \mathbf{e}_\Theta + p_\phi \mathbf{e}_\phi)\frac{f'(r)}{r} = \mathbf{e}_r p_r f''(r) + (\mathbf{p} - p_r \mathbf{e}_r)\frac{f'(r)}{r} \; . \tag{29}$$

Combining this result with the expression for the Laplacian in spherical coordinates, $\nabla^2 f(r) = f''(r) + 2f'(r)/r$ we immediately get

$$\mathbf{J} = D\left(p_r \mathbf{e}_r\left(f''(r) - \frac{f'(r)}{r}\right) - \mathbf{p}\left(f''(r) + \frac{f'(r)}{r}\right)\right) \; , \tag{30}$$

or, by inserting the expression for $f(r)$ and doing the derivatives

$$\mathbf{J} = \frac{1}{2\pi^{3/2}r^3}\left(3p_r\mathbf{e}_r\left[E(\alpha r) - \alpha r(1 + \frac{2}{3}(\alpha r)^2)e^{-(\alpha r)^2})\right] - \mathbf{p}\left[E(\alpha r) - \alpha r(1 + 2(\alpha r)^2)e^{-\alpha r^2}\right]\right) \; . \tag{31}$$

In the following we specialize first to a vertical dipole source, and then, to a horizontal one. Since the problem is linear, we may obtain any dipole field by combining the two.

The field from a vertical transmitter

A vertical transmitter is defined by $\mathbf{p} = \mathbf{e}_z p$, where $\mathbf{e}_z$ is the unit vector along the z-axis. We will denote the current field from a vertical dipole $\mathbf{J}^v$. The vertical component of the current field $J_z^v = \mathbf{J}^v \cdot \mathbf{e}_z$ is obtained by noting that

$$p_r \mathbf{e}_r \cdot \mathbf{e}_z = p\cos^2\Theta = p\frac{z^2}{r^2} \tag{32}$$

so that Eq. (32) gives

$$J_z^v = \frac{p}{2\pi^{3/2}r^3}\left(\left[\left(1 - \frac{3z^2}{r^2}\right) + 2\left(1 - \frac{z^2}{r^2}\right)(\alpha r)^2\right](\alpha r)e^{-(\alpha r)^2} - \left[1 - \frac{3z^2}{r^2}\right]E(\alpha r)\right) \; . \tag{33}$$

The horizontal component J_R^v is obtained by noting that the problem has cylindrical symmetry around the axis of the dipole, and therefore $J_R^v = J_x^v$ at $\phi = 0$. Using the relations $\mathbf{e}_r \cdot \mathbf{e}_x = R/r$, $\mathbf{e}_x \cdot \mathbf{p} = 0$ and the expression for p_r we get

$$J_R^v = \frac{pzR}{2\pi^{3/2}r^5}\left(3E(\alpha r) - (3 + 2(\alpha r)^2)(\alpha r)e^{-\alpha r^2}\right) \ . \tag{34}$$

The field from a horizontal transmitter

Since we are only interested in the relative angle between $\mathbf{p}$ and the observation point we will align the dipole with the x-axis, $\mathbf{p} = p\mathbf{e}_x$. The relations $\mathbf{p} \cdot \mathbf{e}_z = 0$ and $p_r = \cos\phi p\ R/r$ now gives the vertical component

$$J_z^h = \frac{pzR\cos\phi}{2\pi^{3/2}r^5}\left(3E(\alpha r) - (3 + 2(\alpha r)^2)(\alpha r)e^{-(\alpha r)^2}\right) = \cos\phi J_R^v \tag{35}$$

where the last relation may also be derived from geometrical considerations.

A horizontal dipole still has cylindrical symmetry around the dipole axis. But with cylindrical coordinates around the z-axis $\mathbf{J}^h$ has both angular and radial components.

The radial component is easily derived. The relations $\mathbf{p} \cdot \mathbf{e}_R = p\cos\phi$ and $\mathbf{e}_R \cdot \mathbf{e}_r = z/r$ give

$$J_R^h = \frac{3p\cos\phi}{2\pi^{3/2}r^3}\left(\left(\frac{2}{3} - \frac{z^2}{r^2}\right)E(\alpha r) - \left(\frac{2}{3} - \left(1 + \frac{2}{3}(\alpha r)^2\right)\frac{z^2}{r^2}\right)(\alpha r)e^{-(\alpha r)^2}\right) \ . \tag{36}$$

The angular component is obtain through the use of $\mathbf{e}_\phi \cdot \mathbf{p} = -p\sin\phi$ and $\mathbf{e}_\phi \cdot \mathbf{e}_r = 0$ and may be written

$$J_\phi^h = \frac{p\sin\phi}{2\pi^{3/2}r^3}\left(E(\alpha r) - (1 + 2(\alpha r)^2)(\alpha r)e^{-(\alpha r)^2}\right) \ . \tag{37}$$

Taylor series for the current components

It is fairly straight forward to obtain a closed form expression for the Taylor coefficients of the various current components. From these the asymptotic, long time approximations to the fields follow directly.

First, note that the above equations may be summarized on the form

$$\begin{aligned} J_z^v &= \frac{p}{2\pi^{3/2} r^3} v_z(\alpha r) \\ J_R^v &= \frac{3pzR}{2\pi^{3/2} r^5} v_R(\alpha r) \\ J_z^h &= \frac{3pzR\cos\phi}{2\pi^{3/2} r^5} v_R(\alpha r) \\ J_R^h &= \frac{3p\cos\phi}{2\pi^{3/2} r^3} h_R(\alpha r) \\ J_\phi^h &= \frac{p\sin\phi}{2\pi^{3/2} r^3} h_\phi(\alpha r) \end{aligned} \tag{38}$$

where

$$\begin{aligned} v_z(x) &= \left(1 - 3\frac{z^2}{r^2} + 2x^2(1 - \frac{z^2}{x^2})\right) xe^{-x^2} - (1 - 3\frac{z^2}{r^2})E(x) \\ v_R(x) &= E(x) - \left(1 + \frac{2}{3}x^2\right) xe^{-x^2} \\ h_R(x) &= \left(\frac{2}{3} - \frac{z^2}{r^2}\right) E(x) + \left(\left(1 + \frac{2}{3}x^2\right)\frac{z^2}{r^2} - \frac{2}{3}\right) xe^{-x^2} \\ h_\phi(x) &= E(x) - (1 + 2x^2)xe^{-x^2}\ . \end{aligned} \tag{39}$$

The large x limit, which corresponds to short times, is obtained by inspection, as it amounts to the replacements $xe^{-x^2} \to 0$ and $E(x) \to \sqrt{\pi}/2$, giving the current components as the standard dipole polynomials.

For instance, at small t, i.e. $x \gg 1$, $j_z^v = -p(1 - 3(z/r)^2)/(4\pi r^3)$, which has exactly the same form as the vacuum electric field around a charge dipole of dipole moment p. The reason for this is that a dipole is a linear superposition of two monopole field. Both in the vacuum case and the case of the conductive medium the monopole fields are spherical symmetric and satisfy $\nabla\cdot\mathbf{E} = 0$. Hence, one field is obtained from the other by the replacements $qa \to Ia = p$.

The functions of Eqs. (40) all share the feature that the terms of lowest order in x vanish. Let us take $h_\phi(x)$ as an example. By using the Taylor series

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \text{ and } E(x) = \int_0^x dy\ e^{-y^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \tag{40}$$

we can write

$$h_\phi(x) = E(x) - (1+2x^2)xe^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}\left(\frac{1}{2n+1} - 1 - 2x^2\right) = -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}\left(\frac{2n}{2n+1} + 2 \tag{41}$$

where we have grouped the last two terms. Now,

$$\sum_{n=0}^{\infty} a_n x^{2n} = x^2 \sum_{n=0}^{\infty} a_{n+1} x^{2n} \tag{42}$$

if only $a_0 = 0$. Using this relation in the above sum for $h_\phi(x)$ we obtain

$$h_\phi(x) = -4x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \left(\frac{n+1}{2n+3} \right) . \tag{43}$$

Similarely, we get the other functions as

$$\begin{aligned} v_z(x) &= 4x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \left(\frac{1+n(R/r)^2}{2n+3} \right) \\ v_R(x) &= \frac{4x^5}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \left(\frac{1}{2n+5} \right) \\ h_R(x) &= -\frac{2x^3}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \left(\frac{1-2n(z/r)^2}{2n+3} \right) \end{aligned} \tag{44}$$

using the same technique.

The fields in a homogeneous halfspace

In order to get a first approximation of the current field in a conducting sea we now consider a halfspace. The corresponding sea would be infinitely deep, but the qualitative corrrespondence between the sea plus sea bottom, and an infinitely deep sea is still good.

The boundary condition on the sea surface where $z = 0$ is that vertical current component vanishes, i.e. $J_z = 0$. For simplicity we shall consider the sea to be at $z > 0$ in the upper halfplane. Also we shall only be concerned with the expressions for the asymptotic, long- and short time approximations to the fields. However, the full expressions are readily available from the below theory.

The boundary condition may be realized by introducing mirror dipoles $\mathbf{p}'$ in the lower halfspace, as is illustrated in Fig. 3. This mirror dipoles create the correct field in the upper halfspace, but not in the lower (non-conductive) lower halfspace.

Note that the horizontal mirror dipole have the same direction as the real dipole, while the vertical mirror dipole is oriented in the opposite direction. This means that far away the horizontal dipole will be reinforced by the boundary condition, while the vertical dipole will

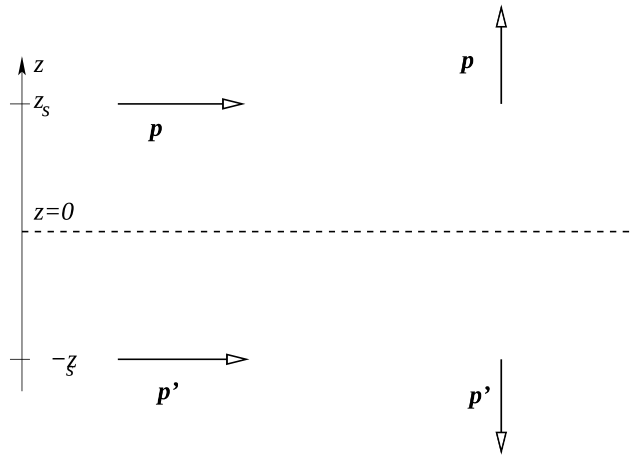

Figure 3: Mirror dipoles $\mathbf{p}'$ that create the proper $J_z = 0$ boundary condition. The dipoles are located at $z = z_s$ and the mirror dipoles are located at $z = -z_s$.

tend to be cancelled by its mirror, and hence strongly weakened by the boundary condition. Mathematically, the halfspace field due to a horizontal dipole takes the form

$$\mathbf{j}^h(R, z, t) = \mathbf{J}^h(R, z - z_s, t) + \mathbf{J}^h(R, z + z_s, t) \tag{45}$$

while the field due to a vertical dipole has the form

$$\mathbf{j}^v(R, z, t) = \mathbf{J}^v(R, z - z_s, t) - \mathbf{J}^v(R, z + z_s, t)\ . \tag{46}$$

With the expressions we have derived it is a straightforward matter to plot the various current components. In Fig. 4 $\log_{10}|j_z^v(t)|$ is plotted against time. The signchange takes place at t =1s, or approximately where $\alpha = 1$.This is just as expected, according to the qualitative interpretation of Fig.1.

In the following we will use the large time approximations ($t \gg r^2/(4D)$) obtained from Eqs. (40) in the $x \ll 1$ limit. The smallest nonvanishing series-appoximations for j_z^v and j_z^v are given by $n = 1$ for the j_z^v case, and $n = 0$ for j_z^v. The results are

$$\begin{aligned} j_z^v(R, z, t) &\approx \frac{2p\alpha^5}{5\pi^{3/2}}((z + z_s)^2 - (z - z_s)^2) = \frac{8p\alpha^5}{5\pi^{3/2}} z z_s \\ j_R^v(R, z, t) &\approx -\frac{4p\alpha^5}{5\pi^{3/2}} R z_s \end{aligned} \tag{47}$$

which means that the current due to a vertical dipole decays as $J^v \propto 1/t^{5/2}$ for large t. Note that $j_z^v(R, z, t) > 0$, which means that the late vertical current has the same sign as the dipole itself. At $t = 0$ the directions are opposite, which is again the signchange.

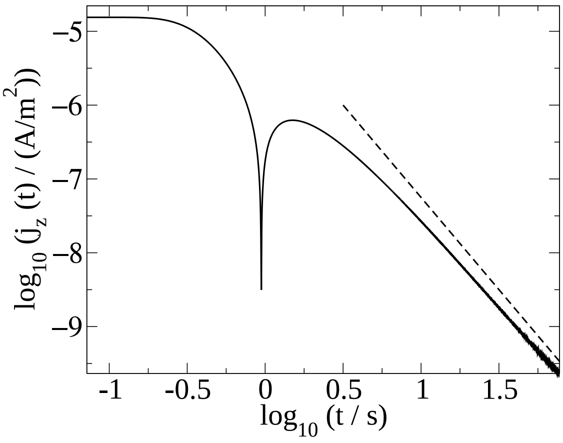

Figure 4: The absolute value of the vertical current at depth $z = 200$m and offset $R = 1000$m due to a vertical dipole of current dipole moment $p = 600000$Am at $z_s = 100$m. The stapled line shows the expected asymptotic behavior $j_z \propto 1/t^{5/2}$.

For late times the horizontal fields are given as

$$\begin{aligned} j_z^h(R,z,t) &\approx \frac{4p\cos\phi\alpha^5}{5\pi^{3/2}} zR \\ j_R^h(R,z,t) &\approx -\frac{2p\cos\phi\alpha^3}{3\pi^{3/2}} \\ j_\phi^h(R,z,t) &\approx -\frac{4p\sin\phi\alpha^3}{3\pi^{3/2}} \end{aligned} \tag{48}$$

which shows that the current due to a horizontal dipole decays as $J^v \propto 1/t^{3/2}$.

Tilt effects

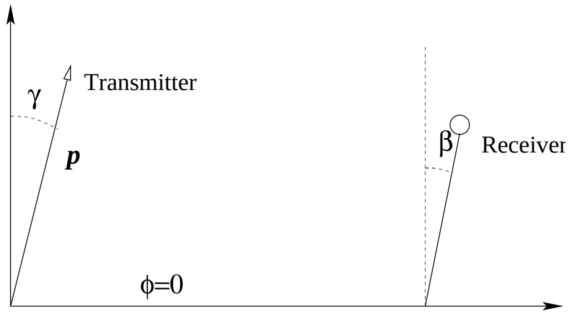

Figure 5: A transmitter-receiver setup with tilt in the same plane.

Now that we have the contributions from the horizontal and vertical dipole components it is an easy matter to combine them into a dipole of arbitrary orientation. Figure 5 shows a situation where the transmitter and receiver are tilted in the same plane. The current j along the receiver, which corresponds to the electric field measured there, is

$$j = \cos\beta j_z + \sin\beta j_R \tag{49}$$

where

$$\begin{aligned} j_z &= \cos\gamma j_z^v + \sin\gamma j_z^h \\ j_R &= \cos\gamma j_R^v + \sin\gamma j_R^h \, . \end{aligned} \tag{50}$$

We may also calculate the effect of orthogonal transmitter and receiver tilts through the j_ϕ^h component. Since $\sin\phi = 0$ for $\phi = 0$ the orthogonal tilt effects are also zero. If the angles are small we can use the first order approximations to the sines and cosines in Eqs. (51) to get

$$j \approx j_z^v + \beta j_R^v + \gamma j_z^h + \beta\gamma j_R^h = j_z^v(1 + \beta\frac{j_R^v}{j_z^v} + \gamma\frac{j_z^h}{j_z^v} + \beta\gamma\frac{j_R^h}{j_z^v}) \, . \tag{51}$$

In the above expression j_z^v is the desired signal and the last three terms represent corrections to this. In order to recognize tilt effects in measured field responses it is instructive to plot the above function for various angles. Figure 6 shows the effect of the various terms in Eq. (52) when the two tilt angles are 4^o. In this figure the exact expressions of Eqs. (40) have been used. Note that a finite receiver tilt, $\beta = 4^o$ without a transmitter tilt, produces a shifted time for the sign-change, but otherwise a curve which is very similar to the fully vertical case. The opposite situation with a finite transmitter tilt $\gamma = 4^o$ but no receiver tilt, also produces a shift in the sign-change. The direction of the shift dpends on the sign og β and γ. Finally when both β and γ are finite we see that the asymptotic $1/t^{5/2}$ behavior is replaced by the dominant $1/t^{3/2}$ behavior.

Using the late time approximations Eqs. (49) in Eq. (52) gives

$$j = j_z^v(1 - \beta\frac{R}{2z} + \gamma\frac{R}{2z_s} + \beta\gamma\frac{5}{12\alpha^2 z z_z}) \, . \tag{52}$$

which shows the correction terms are all potentially large. In order for each of the corrections

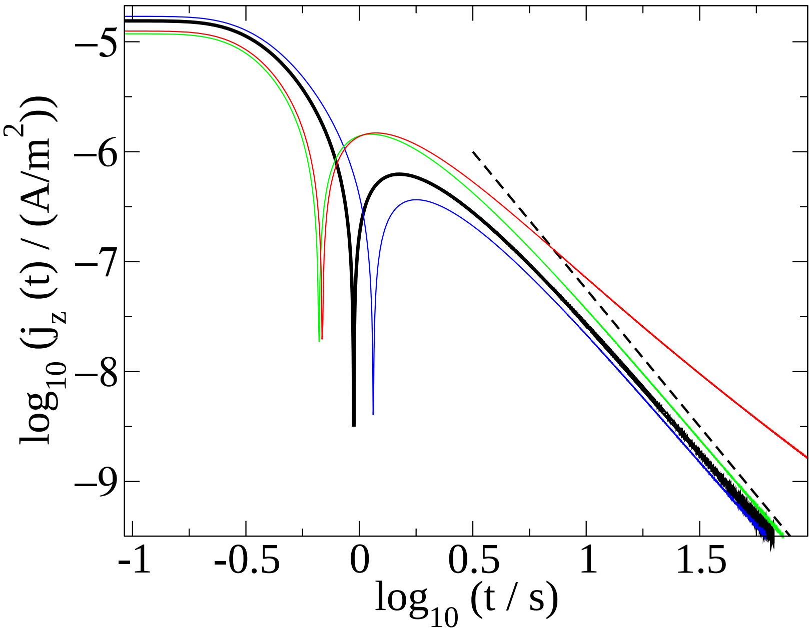

Figure 6: The receiver signal (measured as the current density along the receiver) as a function of time when $\gamma = \beta = 0$ (black) $\gamma = 4^o,\ \beta = 0$ (green) $\gamma = 0,\ \beta = 4$ (blue) and $\gamma = 4^o,\ \beta = 4$ (red). The stapled line shows the expected asymptotic behavior $j_z \propto 1/t^{5/2}$, and the parameter values are as in Fig. 4.

to j to be within some tolerance ϵ we need to make the conditions

$$\begin{aligned}
\beta\frac{R}{2z} &< \epsilon \\
\gamma\frac{R}{2z_s} &< \epsilon \\
\beta\gamma\frac{5}{12\alpha^2 z z_z} &= \beta\gamma\frac{5t}{3\sigma\mu z z_s} < \epsilon\ .
\end{aligned} \tag{53}$$

Focusing on the last of the above conditions first, and taking $t = 10$s, $\sigma = 2S/m$, which is the conductivity of saltwater saturated soil, rather than seawater, $z_s = z/2 = 150$m we get the condition

$$\beta\gamma < 5\ 10^{-3}\epsilon\ . \tag{54}$$

If ϵ =5 % and $\beta = \gamma$ we find a tolerance of $\beta = 1^o$, which is consistent with the numerical estimates of Boris. Focusing on the first two conditions, and using an offset of $R = 3$km we get

$$\begin{aligned} \beta &< \frac{2z\epsilon}{R} = 0.01 \approx 0.5^o \\ \gamma &< \frac{z\epsilon}{R} = 0.005 \approx 0.25^o \ , \end{aligned} \qquad (55)$$

which gives a slightly stronger condition on receiver verticality. The last criterion is rather tough to fullfill. If we denote the uncertainty in the boat position ΔR and $\gamma = \Delta R/R$, it takes the form

$$\Delta R < \frac{z^2\epsilon}{R} \ , \qquad (56)$$

and we see that the tolerance in the boat position increases as the square of the depth. For ϵ =5 % , R = 3000m and the depth z = 500m, ΔR = 4m, which seems practical. At smaller depths, however, the acceptable offset decreases quickly. Remember that the initial assumption of a point dipole implies that our solutions are inaccurate when $R < z$.