Theory for Ken Tores experiments

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(Dated: 10th December 2007)

We obtain the scaling behaviour of the cluster probability distribution and the relation between permeability and flow rate.

PACS numbers: 82.70.Dd; 75.50.Mm; 05.40.-a; 83.10.Pp; 83.80.Gv

I. THEORY

In the following we shall formulate a simple and minimal scaling theory for the purpose of predicting the exponents characterizing the cluster probability distribution and the permeability. For this purpose we shall make order of magnitude estimates for the key quantites at hand, and disregard less relevant prefactors.

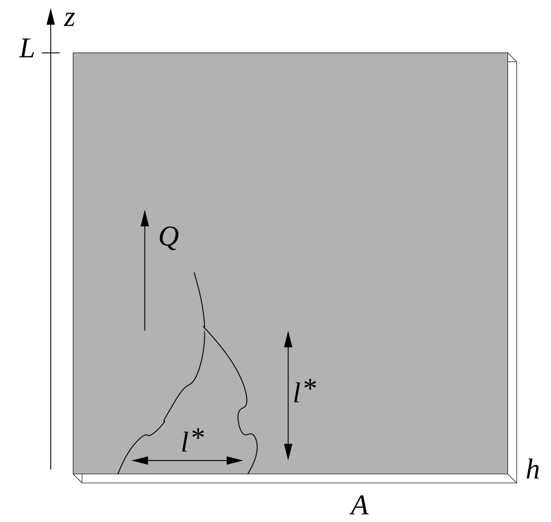

Figure 1: Cartoon of the pathway formed by the non-wetting fluid and some system length scales. The inlet cross section area is A and the cell thickness is h.

A. Characteristic length scale

The imposed flow rate will determine the characteristic cluster length scale that is observed experimentally. It will be useful to formulate everything in terms of this characteristic length scale, and we shall take this scale to be the critical extent in the z-direction l^* below which air clusters become immobile. Note that the same length determines when clusters become unstable against breakup, since the mechanism of cluster movement is the same as that of cluster fragmentation. The permeability is defined as

$$\kappa(Q) = \mu \frac{QL}{A\Delta P_L} , \tag{1}$$

where ΔP_L is the pressure drop across the entire length L of the system. and shall assume that the cluster dynamics is governed by the magnitude of the average pressure gradient throughout the liquid phase. The condition for a cluster to move is that the viscous pressure drop in the surrounding fluid exceeds the cappilary threshold pressures in the cluster. This threshold pressure, which is given by the sum, or rather difference, of the imbibition and drainage pressures in opposing sides of the moving cluster, we denote P_t. The critical mobility length is linear in P_t, which is a fluctuating quantity, and we shall define l^* as the average mobility lenght. The condition that defines l^* may then be formulated in the following way

$$\Delta P_{l^*} = \overline{P}_t , \tag{2}$$

where $\overline{P}_t$ is the average of P_t. Making the mean field assumption that the pressure gradient is constant, $\Delta P_{l^*} = \Delta P_L l^*/L$, and we can use Eq. (1) to write

$$l^* = \frac{\kappa A \overline{P}_t}{\mu Q} , \tag{3}$$

which is the desired result.

B. Dissipation and the permeability exponent 1/2

Due to the fluctuations in pore capillary thresholds the liquid pathways through the system will locally be directed in some random angle relative to the mean flow direction. For this reason the characteristic cluster width will depend on the characteristic length of the clusters, and for simplicity we shall assume that the characteristic width equals the characteristic length l^*, as is illustrated in the figure.

Moreover, direct observation of the experiments indicate that most of the wetting flow takes place in narrow channels defined at some regular spacing. The motion and configuration of the air clusters seem to show that the width of the channels is a pore size or two, and that the permeability in between channels is made relatively

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low by the presence of the air clusters. Taking the spacing between channels as l^* we may then define a mobile volume

$$V_{mob} = ahL\frac{L}{l^*} \tag{4}$$

where a is the pore width, and the last fraction is the number of channels through system.

The total dissipation D_f in the wetting fluid may be written as

$$D_f = -\int_{V_{mob}} dVu\nabla P = \frac{\mu}{\kappa_0}\int_{V_{mob}} dVu^2 \tag{5}$$

where we have applied Darcys law, the integration is taken over the mobile part of the non-wetting fluid, and therefore κ_0 is the single-phase permeablity (the integration is done in between the air clusters). Taking the local Darcy velocity as a constant, $u = (Q/A)(V/V_{mob})$ we get

$$D_f = \frac{\mu Q^2 V^2}{\kappa_0 A^2 V_{mob}} \tag{6}$$

upon integration over V_{mob}. Using Eq. (4) gives us

$$D_f = \frac{\mu Q^2 V l^*}{\kappa_0 A^2 a} \; . \tag{7}$$

Now, since the overall interface area between the wetting and non-wetting phase can be taken as a constant in steady state, the work rate done through the external pressure drop must equal the dissipation,

$$Q\Delta P_L = D_f \; . \tag{8}$$

By writing $\Delta P_L = \overline{P}_t L/l^*$ in this equation and inserting the above expression for D_f we immediately get

$$l^{*2} = \frac{\overline{P}_t a\kappa_0 A}{\mu Q} \tag{9}$$

where we have also used that $V/A = L$. This equation should be compared to the definition of Eq. (3). If we square Eq. (3), insert the result on the left hand side of Eq. (9), and solve for $\kappa(Q)$, we get the two phase permeability

$$\kappa = \kappa_0^2 \frac{Q\mu a}{A\kappa_0\overline{P}_t} = \kappa_0^2 \mathrm{Ca} \tag{10}$$

where the capillary number

$$\mathrm{Ca} = \frac{Q\mu a}{A\kappa_0\overline{P}_t} \tag{11}$$

reduces to the old definition when $\overline{P}_t \to \gamma/a$. This shows that $\kappa \propto Q^{1/2}$, as is pretty much observed experimentally.

C. The flux boundary condition and the distribution exponent $\beta = 3$

Now, lets turn to the distribution of clusters. Assuming that there is only one relevant length scale l^* in the system, we are justified in writing the cluster size distribution as

$$P(l) = l^{-\beta} h(l/l^*) \tag{12}$$

where β is the exponent to be determined. Note that this scaling form should only be expected to hold for large l. For small l there is another scale, the pore scale a, that is important. In fact, if the above scaling form were to hold for all l, normalization would give

$$1 = \int_0^\infty dl l^{-\beta} h(l/l^*) = l^{*(1-\beta)} \int_0^\infty dx h(x) \tag{13}$$

where $x = l/l^*$. Since the right hand side must be l^*-independent, we would get that $\beta = 1$.

We shall however, only assume the above scaling form for $l > l^*$ and then normalization does not give a condition to fix β. The condition we may use is the boundary condition that the air flow rate equals the flow rate of the wetting fluid. This implies that

$$Q = \frac{\mathcal{N}}{L}\int dl l^2 P(l) V(l) \tag{14}$$

where $V(l)$ is the average centre of mass velocity of an air cluster of linear extent l, and $\mathcal{N}$ is the total number of clusters. This number is measured experimentally and found to depend only weakly on Q.

To obtain the velocity $V(l)$ we assume that the cluster displacement happens in discrete steps of pore-pair motion, i.e. that one pore is invadeded by wetting fluid in the back of the cluster and another pore by air in the front of the cluster. Taking these pores to have a separation l and the displacement at the pore level to have a magnitude a, the center-of-mass velocity satisfies

$$l^2 V(l) = val \tag{15}$$

where v is the average meniscus speed through the pores. Since only clusters of length $l > l^*$ are able to move, $v \propto \Theta(l/l^* - 1)$, where $\Theta(l/l^* - 1)$ is the Lorentz-Heaviside function. We shall also assume that once the capillary threshold is exceeded, the menisci move at the velocity of the surrounding wetting fluid. This is consistent with the picture of strongly constrained fluids in mobile channels, where there is not a lot of freedom for one fluid to pass the other.

Reverting to our mean field assumption that the pressure gradient is constant we then write

$$v = \frac{-\kappa_0\overline{\nabla P}}{\mu}\Theta(l/l^* - 1) = \frac{-\kappa_0\overline{P}_t}{l^*\mu}\Theta(l/l^* - 1) \; , \tag{16}$$

or by insertion in Eq. (15)

$$V(l) = \frac{1}{l^{*2}}\frac{\kappa_0\overline{P}_t a}{\mu}\frac{l^*}{l}\Theta(l/l^* - 1) \; . \tag{17}$$

Note that by substitution of l^* from Eq. (9) the above equation takes the simple form

$$V(l) = \frac{Q}{A} f(l/l^*) \tag{18}$$

where $f(x) = \Theta(x-1)/x$. Inserting this expression for $V(l)$ in Eq. (14) we get

$$Q = \frac{\mathcal{N}}{L} \int_{l^*}^{\infty} dl l^2 P(l) \frac{Q}{A} f(l/l^*) \tag{19}$$

in which Q cancels and leaves us with the expression

$$1 = l^{*(3-\beta)} \frac{\mathcal{N}}{AL} \int_{l^*}^{\infty} dx x^{2-\beta} f(x) h(x) \; . \tag{20}$$

Since the right hand side must be l^*-independent, we get

$$\beta = 3 \; . \tag{21}$$

Note that by changing variables to the cluster area $s = l^2$, the probability distribution $H(s)$ is given by the requirement that

$$ds H(s) = dl P(l) \tag{22}$$

or

$$H(s) = P(l(s)) \frac{dl}{ds} = s^{-(\beta+1)/2} g(s/s^*) = s^{-2} g(s/s^*) \; , \tag{23}$$

where $s^* = l^{*2}$. Note also that, if we multiply the above g-function by s^{*2}/s^{*2} the above scaling form may be written

$$H(s) = s^{*-2} \hat{g}(s/s^*) \propto Q^2 \hat{g}(s/s^*) \; , \tag{24}$$

which is our final result.